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Social Shaping of Dynamic Multi-Agent Systems over a Finite Horizon

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Abstract— **This paper studies self-sustained dynamic multi-agent systems (MAS) for decentralized resource allocation operating at a competitive equilibrium over a finite horizon. The utility of resource consumption, along with the income from resource exchange, forms each agent's payoff which is aimed to be maximized. Each utility function is parameterized by individual preferences which can be designed by agents independently. By shaping these preferences and proposing a set of utility functions, we can guarantee that the optimal resource price at the competitive equilibrium always remains socially acceptable, i.e., it never violates a given threshold that indicates affordability. First, we show this problem is solvable at the conceptual level under some convexity assumptions. Then, as a benchmark case, we consider quadratic MAS and formulate the associated social shaping problem as a multi-agent LQR problem which enables us to propose explicit utility sets using quadratic programming and dynamic programming. Finally, a numerical algorithm is presented for calculating the range of the preference function parameters which guarantee a socially accepted price. Some illustrative examples are given to examine the effectiveness of the proposed methods.**

I. INTRODUCTION

Control and analysis of multi-agent systems (MAS) have received great attention among researchers due to their wide application in different areas such as economics [1], water systems [2], carbon markets [3], robotics [4], power systems [5], and smart grids [6]. In general, a MAS consists of a group of agents who collaborate through a network to achieve a common objective or compete to reach an individual goal [7]. MAS are capable of solving complex problems in a distributed manner which are much harder or impossible to be solved by a single agent in a centralized way.

One of the most fundamental problems in the literature is efficient resource allocation, which can be addressed by MAS approaches [8], [9]. Depending on the application, there exist two common approaches: (i) social welfare where agents collaborate to maximize the total agents' utilities [10], [11]; (ii) competitive equilibrium in which agents compete to maximize their individual payoffs [12], [17]. In this paper, distributed resource allocation is treated as an optimization problem in which each agent maximizes its payoff under some constraints and the decision variable determines the amount of resource dedicated to each agent. Considering systems with dynamical states, this optimization problem becomes an optimal control problem. A competitive equilibrium, which is the pair of allocated resource and resource price, is proven to be an efficient solution to resource allocation problems by clearing the market [13].

A fundamental theorem in classical welfare economics states that the competitive equilibrium is Pareto optimal, meaning that no agent can deviate from the equilibrium to achieve more profit without reducing other's payoff [14]–[16]. It is also proved that under some convexity assumptions, the competitive equilibrium maximizes social welfare [17]–[19]. Mechanism design is a well-known approach for social welfare maximization [20]. For instance, the Groves mechanism maximizes social welfare in a way that truth-telling of personal information by all agents forms a dominant strategy [21]. The key point in achieving competitive equilibrium is efficient resource pricing that depends on the utility of each agent. The corresponding price, however, is not guaranteed to be affordable for all agents. If some participants select their utility functions aggressively, the price potentially increases to the point that it becomes unaffordable to other agents who have no alternative but to leave the system. In such cases, the available resources are consumed by a limited number of affluent agents, which is not socially fair in societies where it is deemed that all entities are entitled to equal access [22], [23]. A recent example is the Texas power outage disaster in February 2021, when some citizens who had access to electricity during the power outage received outstanding electricity bills for their daily power usage, resulting in the dissatisfaction of customers [24].

In this paper, we investigate how socially acceptable resource pricing at a competitive equilibrium is achievable for self-sustained dynamic MAS with distributed resource allocation over a finite horizon. Agents allocate their resources in a way that their payoff, which consists of the utility from resource consumption and the income from resource exchange, is maximized over the whole horizon. The utility functions selected by agents affect resource pricing at the competitive equilibrium. By parameterizing these utility functions considering the preferences of agents and proposing some bounds on the parameters, we control the resource price such that it never exceeds a given threshold, so we achieve affordability. We face an optimization problem and address it from three points of view.

- A conceptual scheme, based on dynamic programming, is presented to show how the social shaping problem is solvable implicitly under some convexity assumptions for general classes of utility functions. Also, it is proved that when the price is positive, the total supply and

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demand are balanced across the network.

- The social shaping problem is reformulated for quadratic MAS, leading to an LQR problem. Solving the LQR problem using quadratic programming and dynamic programming, we propose two explicit sets for the preferences of agents which are proved to be socially admissible; i.e., they lead to socially acceptable resource prices.
- A numerical algorithm based on the bisection method is presented that provides accurate and practical bounds on the preferences of agents, followed by some convergence results.

The rest of the paper is organized as follows. In Section II, we review the multi-agent model, the system-level equilibria, and the concept of social shaping for dynamic MAS. In Section III, we present a conceptual scheme for solving the social shaping problem. In Section IV, we introduce quadratic MAS and the LQR problem that follows. Then, we propose two explicit sets of agents' preferences using optimization methods. In Section V, we present a numerical algorithm which provides accurate bounds on agents' preferences. Finally, Section VI includes some simulation results and Section VII contains conclusions.

Notation: We denote by $\mathbb{R}$ and $\mathbb{R}^{\geq 0}$ the fields of real numbers and non-negative real numbers, respectively. $\mathbf{I}$ is the identity matrix with a suitable dimension. The symbol $\mathbb{1}$ represents a vector with an appropriate dimension whose entries are all 1. We use $\|\cdot\|$ to denote the Euclidean norm of a vector or its induced matrix norm.

II. Problem Formulation

In this section, we introduce the multi-agent system model, system-level equilibria, and the concept of social shaping.

A. The Dynamic Multi-agent Model

Consider a dynamic MAS with n agents indexed in the set $\mathcal{V} = \{1, 2, \ldots, n\}$. This MAS is studied in the time horizon N. Let time steps be indexed in the set $\mathcal{T} = \{0, 1, \ldots, N-1\}$. Each agent $i \in \mathcal{V}$ is a subsystem with dynamics represented by

$$\mathbf{x}_i(t+1) = \mathbf{A}_i \mathbf{x}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \quad t \in \mathcal{T},$$

where $\mathbf{x}_i(t) \in \mathbb{R}^d$ is the dynamical state, $\mathbf{x}_i(0) \in \mathbb{R}^d$ is the given initial state, and $\mathbf{u}_i(t) \in \mathbb{R}^m$ is the control input. Also, $\mathbf{A}_i \in \mathbb{R}^{d \times d}$ and $\mathbf{B}_i \in \mathbb{R}^{d \times m}$ are fixed matrices. Upon reaching the state $\mathbf{x}_i(t)$ and employing the control input $\mathbf{u}_i(t)$ at time step $t \in \mathcal{T}$, each agent i receives the utility $f_i(\mathbf{x}_i(t), \mathbf{u}_i(t)) = f(\cdot; \theta_i) : \mathbb{R}^d \times \mathbb{R}^m \mapsto \mathbb{R}$, where $\theta_i \in \Theta$ is a personalized parameter of agent i. The terminal utility achieved as a result of reaching the terminal state $\mathbf{x}_i(N)$ is denoted by $\phi_i(\mathbf{x}_i(N)) = \phi(\cdot; \theta_i) : \mathbb{R}^d \mapsto \mathbb{R}$. At each time step $t \in \mathcal{T}$, agent i provides a local energy supply $a_i(t) \in \mathbb{R}^{\geq 0}$, and consumes an amount of energy $h_i(\mathbf{u}_i(t)) : \mathbb{R}^m \mapsto \mathbb{R}^{\geq 0}$ as a result of taking the control action $\mathbf{u}_i(t)$. The overall network supply $C(t) > 0$ is then defined as $C(t) := \sum_{i=1}^n a_i(t)$ for $t \in \mathcal{T}$. Agents are interconnected through a network to sell (or buy) their surplus (or shortage) of energy. This means each agent $i \in \mathcal{V}$ can decide how much of their extra energy $a_i(t) - h_i(\mathbf{u}_i(t))$ is going to be traded through the network leading to a new decision variable named *strategic trading decision*, denoted by $e_i(t) \in \mathbb{R}$. There is a physical constraint indicating the traded resource for each agent can never be greater than the surplus of resource, i.e., $e_i(t) \leq a_i(t) - h_i(\mathbf{u}_i(t))$. The price for unit resource exchange across the network at each time step $t \in \mathcal{T}$ is denoted by λ_t. Then, the income or cost from resource exchange for agent i is represented by $\lambda_t e_i(t)$.

B. System-level Equilibria

Let $\mathbf{U}_i = (\mathbf{u}_i^\top(0), \ldots, \mathbf{u}_i^\top(N-1))^\top$ and $\mathbf{E}_i = (e_i(0), \ldots, e_i(N-1))^\top$ denote the vector of control inputs and the vector of strategic trading decisions associated with agent i over the whole time horizon, respectively. Also, let $\mathbf{u}(t) = (\mathbf{u}_1^\top(t), \ldots, \mathbf{u}_n^\top(t))^\top$ and $\mathbf{e}(t) = (e_1(t), \ldots, e_n(t))^\top$ denote the vector of control inputs and the vector of strategic trading decisions associated with all agents at time step $t \in \mathcal{T}$, respectively. Let $\mathbf{U} = (\mathbf{u}^\top(0), \ldots, \mathbf{u}^\top(N-1))^\top$ and $\mathbf{E} = (\mathbf{e}^\top(0), \ldots, \mathbf{e}^\top(N-1))^\top$ be the vector of all control inputs and the vector of all strategic trading decisions at all time steps, respectively. Let $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{N-1})^\top$ denote the vector of resource prices throughout the entire time horizon.

Definition 1: The *competitive equilibrium* for a dynamic MAS is the triplet $(\boldsymbol{\lambda}^*, \mathbf{U}^*, \mathbf{E}^*)$ which satisfies the following two conditions.

(i) Given $\boldsymbol{\lambda}^*$, the pair $(\mathbf{U}^*, \mathbf{E}^*)$ maximizes the individual payoff function of each agent; i.e., each $(\mathbf{U}_i^*, \mathbf{E}_i^*)$ solves the following constrained maximization problem

$$\begin{aligned} \max_{\mathbf{U}_i, \mathbf{E}_i} \quad & \phi(\mathbf{x}_i(N); \theta_i) + \sum_{t=0}^{N-1} \Big(f(\mathbf{x}_i(t), \mathbf{u}_i(t); \theta_i) + \lambda_t^* e_i(t) \Big) \\ \text{s.t.} \quad & \mathbf{x}_i(t+1) = \mathbf{A}_i \mathbf{x}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \\ & e_i(t) \leq a_i(t) - h_i(\mathbf{u}_i(t)), \quad t \in \mathcal{T}. \end{aligned} \tag{1}$$

(ii) The optimal strategic trading $\mathbf{E}^*$ balances the total traded resource across the network at each time step; that is,

$$\sum_{i=1}^n e_i^*(t) = 0, \quad t \in \mathcal{T}. \tag{2}$$

Definition 2: The *social welfare equilibrium* for a dynamic MAS is the pair $(\mathbf{U}^*, \mathbf{E}^*)$ which solves the following optimization problem

$$\begin{aligned} \max_{\mathbf{U}, \mathbf{E}} \quad & \sum_{i=1}^n \left(\phi(\mathbf{x}_i(N); \theta_i) + \sum_{t=0}^{N-1} f(\mathbf{x}_i(t), \mathbf{u}_i(t); \theta_i) \right) \\ \text{s.t.} \quad & \mathbf{x}_i(t+1) = \mathbf{A}_i \mathbf{x}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \\ & e_i(t) \leq a_i(t) - h_i(\mathbf{u}_i(t)), \\ & \sum_{i=1}^n e_i(t) = 0, \quad t \in \mathcal{T}, \ i \in \mathcal{V}. \end{aligned} \tag{3}$$

In the social welfare equilibrium, the total agent utility functions are maximized.

Assumption 1: $f(\cdot;\theta_i)$ and $\phi(\cdot;\theta_i)$ are concave functions for all $i \in \mathcal{V}$. Additionally, $h_i(\cdot)$ is a non-negative convex function such that $h_i(\mathbf{z}) < b$ represents a bounded open set of $\mathbf{z}$ in $\mathbb{R}^m$ for $b > 0$ and $i \in \mathcal{V}$. Furthermore, assume $\sum_{i=1}^n a_i(t) > 0$ for all $t \in \mathcal{T}$.

Proposition 1 (as in [17]): Suppose Assumption 1 holds. Then the competitive equilibrium and the social welfare equilibrium exist and coincide. Additionally, the optimal price λ_t^* in (1) is obtained from the Lagrange multiplier associated with the balancing equality constraint $\sum_{i=1}^n e_i(t) = 0$ in (3).

C. Social Shaping Problem

The optimal price λ_t^*, which is the Lagrange multiplier corresponding to the equality constraint in (3), depends on the utility functions of agents. If there are no regulations on the choice of utility functions, the price may become extremely high and unaffordable for some agents. In this case, those who have found the price unaffordable cannot compete in the market and have no alternative but to leave the system. Consequently, all of the resources will be consumed by a limited number of agents who have dominated the price by aggressively selecting their utilities. This is indeed socially unfair and not sustainable. So we need a mechanism, called social shaping, which ensures the price is always below an acceptable threshold denoted by $\lambda^\dagger \in \mathbb{R}^{>0}$. The problem of social shaping is addressed for static MAS in [22] and [23]. Now, we define an extended version of the social shaping problem for dynamic MAS as follows.

Definition 3 (Social shaping for dynamic MAS): Consider a dynamic MAS whose agents $i \in \mathcal{V}$ have $f(\cdot;\theta_i)$ and $\phi(\cdot;\theta_i)$ as their running utility function and terminal utility function, respectively. Let $\lambda^\dagger \in \mathbb{R}^{>0}$ be the given price threshold accepted by all agents. Find a range Θ of personal parameters θ_i such that if $\theta_i \in \Theta$ for $i \in \mathcal{V}$, or $(\theta_1, \dots, \theta_n) \in \Theta^n$, then we yield $\lambda_t^* \le \lambda^\dagger$ at all time steps $t \in \mathcal{T}$.

III. Conceptual Social Shaping

In this section, we examine how the social shaping problem of dynamic MAS can be solved conceptually.

Lemma 1: Consider the dynamic MAS. If Assumption 1 is satisfied, then $\lambda_t^* \ge 0$ for all $t \in \mathcal{T}$.

Proof: The proof is similar to the proof of Proposition 2 in [17]. ■

Proposition 2: Consider the dynamic MAS. Let Assumption 1 hold. If $\lambda_t^* > 0$ then the total demand and supply are balanced at time step t; that is,

$$\sum_{i=1}^n h_i(\mathbf{u}_i^*(t)) = C(t). \tag{4}$$

Proof: See [25, Proposition 2]. ■

Now, let us work out how the social shaping problem can be solved conceptually. Suppose Assumption 1 holds and $f(\cdot;\theta_i)$, $\phi(\cdot;\theta_i)$, and $h_i(\cdot)$ are continuously differentiable. Then Proposition 1 is satisfied. In this paper, we focus on the competitive optimization problem in (1). According to Lemma 1, there holds $\lambda_t^* \ge 0$. We can skip the case $\lambda_t^* = 0$, because a zero price is always socially resilient. Therefore, it is sufficient to only examine $\lambda_t^* > 0$. Following from Proposition 2, we obtain an equivalent form for the competitive optimization problem in (1) as [25]

$$\begin{aligned} \max_{\mathbf{U}_i} \quad & \phi(\mathbf{x}_i(N);\theta_i) + \sum_{t=0}^{N-1} f(\mathbf{x}_i(t), \mathbf{u}_i(t);\theta_i) \\ & + \sum_{t=0}^{N-1} \lambda_t^* \big[a_i(t) - h_i(\mathbf{u}_i(t))\big] \\ \text{s.t.} \quad & \mathbf{x}_i(t+1) = \mathbf{A}_i \mathbf{x}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \quad t \in \mathcal{T}. \end{aligned} \tag{5}$$

The optimization problem in (5) is in essence an unconstrained optimal control problem which can be solved by the dynamic programming approach. First, introduce the cost-to-go function for agent i from time k to N as

$$\begin{aligned} & J_i^{k \longrightarrow N}(\mathbf{x}_i(k), \mathbf{u}_i(k), \dots, \mathbf{u}_i(N-1), \lambda_k^*, \dots, \lambda_{N-1}^*;\theta_i) \\ & = \phi(\mathbf{x}_i(N);\theta_i) + \sum_{t=k}^{N-1} \Big(f(\mathbf{x}_i(t), \mathbf{u}_i(t);\theta_i) + \lambda_t^* \big[a_i(t) - h_i(\mathbf{u}_i(t))\big] \Big). \end{aligned}$$

Then, the optimal cost-to-go at time k for agent i, which is also called the value function, is represented as

$$\begin{aligned} & V_{i,k}(\mathbf{x}_i(k), \lambda_k^*, \dots, \lambda_{N-1}^*;\theta_i) \\ & = \max_{\mathbf{u}_i(k), \dots, \mathbf{u}_i(N-1)} \quad J_i^{k \longrightarrow N}(\mathbf{x}_i(k), \mathbf{u}_i(k), \dots, \mathbf{u}_i(N-1), \lambda_k^*, \dots, \lambda_{N-1}^*;\theta_i) \\ & \qquad \text{s.t.} \qquad \mathbf{x}_i(t+1) = \mathbf{A}_i \mathbf{x}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \quad t = k, \dots, N-1. \end{aligned}$$

According to the principle of optimality, we obtain

$$\begin{aligned} & V_{i,N}(\mathbf{x}_i(N);\theta_i) = \phi\big(\mathbf{x}_i(N);\theta_i\big), \\ & V_{i,N-1}(\mathbf{x}_i(N-1), \lambda_{N-1}^*;\theta_i) \\ & = \max_{\mathbf{u}_i(N-1)} \quad f(\mathbf{x}_i(N-1), \mathbf{u}_i(N-1);\theta_i) \\ & \qquad + \lambda_{N-1}^* \big[a_i(N-1) - h_i\big(\mathbf{u}_i(N-1)\big)\big] \\ & \qquad + V_{i,N}(\mathbf{A}_i \mathbf{x}_i(N-1) + \mathbf{B}_i \mathbf{u}_i(N-1);\theta_i), \\ & \vdots \\ & V_{i,0}(\mathbf{x}_i(0), \lambda_0^*, \dots, \lambda_{N-1}^*;\theta_i) \\ & = \max_{\mathbf{u}_i(0)} \quad f(\mathbf{x}_i(0), \mathbf{u}_i(0);\theta_i) + \lambda_0^* \big[a_i(0) - h_i\big(\mathbf{u}_i(0)\big)\big] \\ & \qquad + V_{i,1}(\mathbf{A}_i \mathbf{x}_i(0) + \mathbf{B}_i \mathbf{u}_i(0), \lambda_1^*, \dots, \lambda_{N-1}^*;\theta_i). \end{aligned}$$

To obtain the optimal control at time step $k \in \mathcal{T}$, the derivative of the associated objective function with respect to $\mathbf{u}_i(k)$ must equal zero. Proposition 1 implies that such an optimal solution exists, although it might not be unique. Without loss of generality, suppose the optimal solution is unique. Then, we can write $\mathbf{u}_i^*(k)$ as a function of $\mathbf{x}_i(0)$ and all λ_t^* where $t \in \mathcal{T}$, parameterized by θ_i; that is,

$$\mathbf{u}_i^*(k) = l_i^k(\mathbf{x}_i(0), \lambda_0^*, \dots, \lambda_{N-1}^*;\theta_i), \quad k \in \mathcal{T}. \tag{6}$$

Substituting (6) into the equality $\sum_{i=1}^n h_i(\mathbf{u}_i^*(t)) = C(t)$ in (4), we achieve

$$\sum_{i=1}^n h_i(l_i^k(\mathbf{x}_i(0), \lambda_0^*, \dots, \lambda_{N-1}^*;\theta_i)) = C(k), \quad k \in \mathcal{T}. \tag{7}$$

We aim to obtain $\boldsymbol{\lambda}^* = (\lambda_0^*, \ldots, \lambda_{N-1}^*)^\top$. According to (7), we have N equations with N variables. Let $\mathbf{x}(k) = (\mathbf{x}_1^\top(k), \ldots, \mathbf{x}_n^\top(k))^\top$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$, and $\mathbf{C} = (C(0), C(1), \ldots, C(N-1))$. According to Proposition 1, there exists $\boldsymbol{\lambda}^*$ which satisfies (7) although it might not be unique. Among different possible prices that satisfy the equilibrium, we consider the maximum one at each time step. In the rest of this paper, by optimal price we mean the maximum possible price associated with a fixed $\boldsymbol{\theta}$ meeting the equilibrium conditions. Solving (7), the optimal price at each time step $k \in \mathcal{T}$ is obtained as

$$\lambda_k^* = g_k(\mathbf{x}(0), \mathbf{C}; \boldsymbol{\theta}), \quad k = 0, \ldots, N-1.$$

Additionally, for different values of agent preferences $\boldsymbol{\theta}$ we would obtain different optimal prices at each time step. Let us define the maximum value of the set of all possible optimal prices at each time step k, when θ_i takes values in the set Θ (or $\boldsymbol{\theta} \in \Theta^n$), as

$$\chi_k^\Theta := \max_{\boldsymbol{\theta} \in \Theta^n} g_k(\cdot; \boldsymbol{\theta}), \quad k = 0, \ldots, N-1.$$

Next, we introduce

$$\mathbf{G}_\Theta := \left(\chi_0^\Theta, \chi_1^\Theta, \ldots, \chi_{N-1}^\Theta\right)^\top.$$

Each element k in the vector $\mathbf{G}_\Theta$ is the maximum value of optimal prices at time step k, when agent preferences are taken from $\boldsymbol{\theta} \in \Theta^n$. We refer the reader to [25] for a more extensive explanation.

Theorem 1: Consider a dynamic MAS. Let Assumption 1 hold. Suppose $f(\cdot; \theta_i)$, $\phi(\cdot; \theta_i)$, and $h_i(\cdot)$ are continuously differentiable. Let $\lambda^\dagger \in \mathbb{R}^{>0}$ represent the given price threshold accepted by all agents. Then any set Θ satisfying $\mathbf{G}_\Theta \leq \lambda^\dagger \mathbb{1}$ ensures that $\lambda_t^* \leq \lambda^\dagger$ for $t \in \mathcal{T}$, and thus, solves the social shaping problem of agent preferences.

IV. Quadratic Social Shaping

In this section, we examine quadratic utility functions for dynamic MAS and explicitly propose two sets of personal parameters which guarantee that the optimal prices at all time steps are socially resilient.

Assumption 2: Consider the dynamic MAS introduced in Section II-A. Let $\theta_i := (\mathbf{Q}_i, \mathbf{R}_i)$, where $\mathbf{Q}_i \in \mathbb{R}^{d \times d}$, $\mathbf{Q}_i = \mathbf{Q}_i^\top > 0$ and $\mathbf{R}_i \in \mathbb{R}^{m \times m}$, $\mathbf{R}_i = \mathbf{R}_i^\top > 0$. Assume for all $i \in \mathcal{V}$ we have

$$\begin{aligned} f(\mathbf{x}_i(t), \mathbf{u}_i(t); \theta_i) &= -\mathbf{x}_i^\top(t)\mathbf{Q}_i\mathbf{x}_i(t) - \mathbf{u}_i^\top(t)\mathbf{R}_i\mathbf{u}_i(t), \\ \phi(\mathbf{x}_i(N); \theta_i) &= -\mathbf{x}_i^\top(N)\mathbf{Q}_i\mathbf{x}_i(N), \\ h_i(\mathbf{u}_i(t)) &= \mathbf{u}_i^\top(t)\mathbf{H}_i\mathbf{u}_i(t), \end{aligned}$$

where $\mathbf{H}_i \in \mathbb{R}^{m \times m}$, $\mathbf{H}_i = \mathbf{H}_i^\top > 0$.

Assumption 3: Consider the dynamic MAS in Assumption 2 with a given initial state $\mathbf{x}_i(0)$ such that $\|\mathbf{x}_i(0)\| \leq \gamma$, $\|\mathbf{A}_i\| \leq \alpha$, $\|\mathbf{B}_i\| \leq \beta$, and $\mathbf{H}_i \geq \rho\mathbf{I}$ for $i \in \mathcal{V}$. Suppose that $\gamma, \alpha, \beta, \rho \in \mathbb{R}^{>0}$.

We aim to solve the following social shaping problem.

Dynamic & Quadratic Social Shaping Problem. Suppose Assumptions 2 and 3 hold. Let $\lambda^\dagger \in \mathbb{R}^{>0}$ be the given price threshold accepted by all agents, and $\delta_{\max} \in \mathbb{R}^{>0}$ be an upper bound for the norm of the personal parameter $\mathbf{Q}_i$. We propose an admissible set for $\delta_{\max}$ such that all utility functions satisfying $\|\mathbf{Q}_i\| \leq \delta_{\max}$ (or $\mathbf{Q}_i \leq \delta_{\max}\mathbf{I}$) lead to socially acceptable energy prices at all time steps; i.e., $\lambda_t^* \leq \lambda^\dagger$ for $t \in \mathcal{T}$.

To address this problem, we use two approaches: quadratic programming and dynamic programming.

A. Quadratic Programming Approach

Since Assumption 2 is satisfied, Proposition 1 holds. We examine the competitive optimization problem in (1). According to Lemma 1, there holds $\lambda_t^* \geq 0$. We skip the case $\lambda_t^* = 0$, because a zero price is always socially resilient. Hence, it is sufficient to only study $\lambda_t^* > 0$. According to (5), the optimization problem in (1) can be reformulated as

$$\begin{aligned} \max_{\mathbf{U}_i} \quad & -\mathbf{x}_i^\top(N)\mathbf{Q}_i\mathbf{x}_i(N) + \sum_{t=0}^{N-1}\big[-\mathbf{x}_i^\top(t)\mathbf{Q}_i\mathbf{x}_i(t) \\ & -\mathbf{u}_i^\top(t)\mathbf{R}_i\mathbf{u}_i(t) + \lambda_t^*\big(a_i(t) - \mathbf{u}_i^\top(t)\mathbf{H}_i\mathbf{u}_i(t)\big)\big] \\ \text{s.t.} \quad & \mathbf{x}_i(t+1) = \mathbf{A}_i\mathbf{x}_i(t) + \mathbf{B}_i\mathbf{u}_i(t), \quad t \in \mathcal{T}. \end{aligned} \tag{8}$$

Theorem 2: Consider the dynamic MAS described in Assumptions 2 and 3 on the time horizon N. Suppose $\delta_{\max} \in \mathbb{R}^{>0}$ is selected from the following set

$$\mathscr{S}_* = \Bigg\{ \delta_{\max} \in \mathbb{R}^{>0} : \delta_{\max} \sum_{t=k+1}^{N} \Bigg[\gamma\alpha^{2t-k-1} + \beta\sum_{\substack{j=0\\ j\neq k}}^{t-1}\sqrt{\frac{C(j)}{\rho}}\alpha^{2t-j-k-2}\Bigg] \leq \frac{\sqrt{C(k)\rho}}{n\beta}\lambda^\dagger \ \text{ for } \forall k \in \mathcal{T} \Bigg\}.$$

Then for all quadratic utility functions satisfying $\|\mathbf{Q}_i\| \leq \delta_{\max}$ (or $\mathbf{Q}_i \leq \delta_{\max}\mathbf{I}$), the resulting optimal price is socially resilient, i.e., $\boldsymbol{\lambda}^* \leq \lambda^\dagger \mathbb{1}$.

Proof: The proof proceeds based on quadratic programming. See [25, Theorem 2]. ■

B. Dynamic Programming Approach

Similar to Section IV-A, we only consider $\lambda_t^* > 0$ and we deal with the optimization problem in (8) which is equivalent to

$$\begin{aligned} \max_{\mathbf{U}_i} \quad & -\mathbf{x}_i^\top(N)\mathbf{Q}_i\mathbf{x}_i(N) + \sum_{t=0}^{N-1}\big[-\mathbf{x}_i^\top(t)\mathbf{Q}_i\mathbf{x}_i(t) \\ & -\mathbf{u}_i^\top(t)\Big(\mathbf{R}_i + \lambda_t^*\mathbf{H}_i\Big)\mathbf{u}_i(t) + \lambda_t^* a_i(t)\big] \\ \text{s.t.} \quad & \mathbf{x}_i(t+1) = \mathbf{A}_i\mathbf{x}_i(t) + \mathbf{B}_i\mathbf{u}_i(t), \quad t \in \mathcal{T}. \end{aligned}$$

Theorem 3: Consider the dynamic MAS on the time horizon N. Let Assumptions 2 and 3 hold. Suppose $\delta_{\max} \in \mathbb{R}^{>0}$

is selected from the following set

$$\mathscr{S}_* = \Bigg\{ \delta_{\max} \in \mathbb{R}^{>0} : \delta_{\max} \sum_{t=1}^{N} \gamma \alpha^{2t-1} \leq \frac{\sqrt{C(0)\rho}}{n\beta} \lambda^{\dagger}, \\ \delta_{\max} \sum_{t=k+1}^{N} \left[\gamma \alpha^{2t-k-1} + \beta \sum_{j=0}^{k-1} \sqrt{\frac{C(j)}{\rho}} \alpha^{2t-j-k-2} \right] \\ \leq \frac{\sqrt{C(k)\rho}}{n\beta} \lambda^{\dagger} \quad \text{for } \forall k \in \mathcal{T}, k \neq 0 \Bigg\}.$$

Then the resulting $\boldsymbol{\lambda}^*$ is socially resilient for all utility functions satisfying $\|\mathbf{Q}_i\| \leq \delta_{\max}$ (or $\mathbf{Q}_i \leq \delta_{\max}\mathbf{I}$).

Proof: The proof proceeds based on dynamic programming. See [25, Theorem 3]. ■

V. Numerical Algorithm

The two proposed sets in Theorems 2 and 3 are conservative but they give a concrete idea about how the trade-off between utility functions' parameters and the price threshold is achievable. To obtain more accurate and practical results, we propose a numerical algorithm which provides less conservative bounds on the parameters. This algorithm proceeds based on the bisection method.

Numerical Social Shaping Problem. Consider the social welfare problem in (3). Let $\delta_{\max} \in \mathbb{R}^{>0}$ be the designing parameter. Suppose Assumption 2 holds with $\mathbf{Q}_i = q_i\mathbf{I}$ where agents have the freedom to select $q_i \in (0, \delta_{\max}]$. Assume $\lambda^{\dagger}$ is the given price threshold accepted by all agents and $\mathbf{R}_i$ is specified for each $i \in \mathcal{V}$. We aim to find the upper bound $\delta_{\max}$ by a numerical approach such that if $q_i \in (0, \delta_{\max}]$ for $i \in \mathcal{V}$ then $\lambda_t^* \leq \lambda^{\dagger}$ for $t \in \mathcal{T}$. The key steps to reach this purpose are illustrated in Algorithm 1.

Algorithm 1: Bisection-Based Social Shaping

Input: System parameters $\mathbf{A}_i$, $\mathbf{B}_i$, and $\mathbf{H}_i$, the initial state $\mathbf{x}_i(0)$, the time horizon N, the penalty matrix $\mathbf{R}_i$, and the local supply $a_i(t)$ for $i \in \mathcal{V}$ and $t \in \mathcal{T}$.

Structure: Consider $\mathbf{Q}_i = q_i\mathbf{I}$. Define

$$\bar{\lambda}^*(\delta) = \max_{q_1,\ldots,q_n \in (0,\delta]} \max_{t \in \mathcal{T}} \lambda_t^*. \tag{9}$$

Initialize: Set $k = 0$, $b_0 = 0$, and $d_0 = d_\varrho > 0$ such that d_ϱ is sufficiently large to satisfy $\bar{\lambda}^*(d_\varrho) > \lambda^{\dagger}$.

while True **do**
- $L_k = (b_k + d_k)/2, \quad \lambda_k = \bar{\lambda}^*(L_k)$;
- **if** $\lambda_k > \lambda^{\dagger}$ **then**
  - $b_{k+1} = b_k$ and $d_{k+1} = L_k$;
  - $k = k + 1$;
- **else if** $\lambda_k < \lambda^{\dagger}$ **then**
  - $b_{k+1} = L_k$ and $d_{k+1} = d_k$;
  - $k = k + 1$;
- **else**
  - $\delta_{\max} = L_k$;
  - break
- **end**

end

Output: $\delta_{\max} = L_k$ if the algorithm stops after a finite number of steps. Otherwise, $\delta_{\max} = \lim_{k\to\infty} L_k$.

Lemma 2: The function $\bar{\lambda}^*(\delta)$ in (9) is monotonically increasing.

Proof: See [25, Lemma 2]. ■

Theorem 4: The auxiliary variable L_k in Algorithm 1 converges to L^* for some $L^* \in (0, d_\varrho)$ when $k \to \infty$.

Proof: See [25, Theorem 4]. ■

Theorem 5: Suppose there exists $\delta^{\dagger} > 0$ such that $\bar{\lambda}^*(\delta^{\dagger}) = \lambda^{\dagger}$. Then $\delta_{\max}$ obtained from Algorithm 1 satisfies $\bar{\lambda}^*(\delta_{\max}) = \lambda^{\dagger}$.

Proof: See [25, Theorem 5]. ■

VI. Simulation Results

Example 1: Consider a dynamic MAS with 3 agents who satisfy Assumptions 2 and 3 in the time horizon $N = 6$, with the system parameters which are selected as

$$\mathbf{A}_1 = \begin{bmatrix} -0.6 & -0.1 & 0.2 \\ 0.3 & -0.7 & 0.2 \\ 0.2 & -0.3 & 0.8 \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 7 \\ 1 & 6 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0.5 & 0.1 & -0.1 \\ 0.3 & -0.2 & -0.2 \\ -0.2 & 0.3 & -0.3 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 4 & 1 \\ 1 & 6 \\ 3 & 4 \end{bmatrix},$$

$$\mathbf{A}_3 = \begin{bmatrix} -0.4 & 0.2 & 0.2 \\ -0.3 & 0.8 & 0.2 \\ 0.2 & 0.3 & -0.5 \end{bmatrix}, \mathbf{B}_3 = \begin{bmatrix} 9 & 2 \\ 2 & 3 \\ 4 & 5 \end{bmatrix},$$

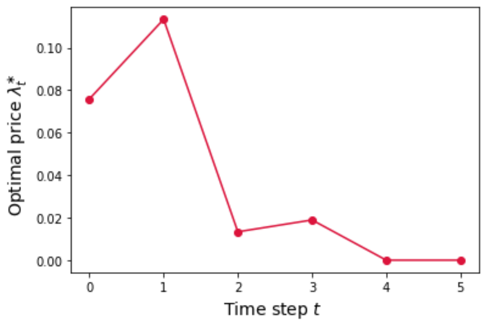
and the initial states $\mathbf{x}_1(0) = (10, 40, 70)^{\top}$, $\mathbf{x}_2(0) = (20, 50, 80)^{\top}$, and $\mathbf{x}_3(0) = (30, 60, 90)^{\top}$. Additionally, suppose agents have the representative local resources $a_1(t) = \sin(\frac{\pi}{6}t) + 1.2$, $a_2(t) = 2\sin(\frac{\pi}{6}t) + 2.2$, and $a_3(t) = 0$. The total network generation is obtained as $C(t) = \sum_{i=1}^{3} a_i(t) = 3\sin(\frac{\pi}{6}t) + 3.4$. Also, let $\lambda^{\dagger} = 30$, $\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{R}_3 = 0.05\mathbf{I}$, and

$$\mathbf{H}_1 = \begin{bmatrix} 5 & 1 \\ 1 & 8 \end{bmatrix}, \mathbf{H}_2 = \begin{bmatrix} 3 & 2 \\ 2 & 7 \end{bmatrix}, \mathbf{H}_3 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

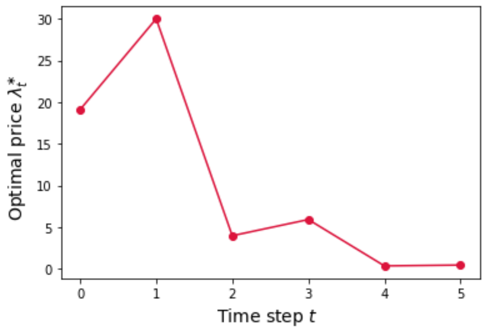
We aim to design $\mathbf{Q}_i$ for $i \in \mathcal{V}$ such that $\lambda_t^* \leq \lambda^{\dagger}$ at all time steps.

Analytical approach: For this example, the upper bound $\delta_{\max}$ of the personalized parameter $\mathbf{Q}_i$ is obtained from Theorems 2 and 3 as 0.00055 and 0.0010, respectively. We observe that Theorem 3 provides a larger upper bound compared to Theorem 2 so we assign $\delta_{\max} = 0.001$. Next, we must select $\mathbf{Q}_i$ such that $\|\mathbf{Q}_i\| \leq \delta_{\max}$ (or $\mathbf{Q}_i \leq \delta_{\max}\mathbf{I}$) for $i \in \mathcal{V}$. Let us carefully choose $\mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{Q}_3 = 0.001\mathbf{I}$. The optimal prices obtained from solving the social welfare problem in (3) are depicted in Fig. 1a at different time steps. As indicated, the optimal prices are much less than 30, and therefore, socially resilient; this confirms that Theorems 2 and 3 are valid but they provide conservative results.

Numerical approach: On the other hand, we run Algorithm 1 for 20 steps with the choice of $d_\varrho = 1$. This value of d_ϱ is sufficiently large that satisfies $\bar{\lambda}^*(d_\varrho) = 148.6 > \lambda^{\dagger}$. The output of the algorithm is $\delta_{\max} = 0.202$. Setting $\mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{Q}_3 = 0.202\mathbf{I}$, the obtained optimal prices are as Fig.



(a) $\delta_{\max} = 0.001$, obtained from Theorem 3.



(b) $\delta_{\max} = 0.202$, obtained from Algorithm 1.

Fig. 1: The optimal price λ_t^* over time steps.

1b, which are less than or equal to 30 and socially resilient. The maximum value of the price throughout the entire time horizon is $\lambda_1^* = 30.0$ happening at time step $t = 1$. If we select $\mathbf{Q}_i = q_i\mathbf{I}$ such that $q_i > 0.202$ for $i \in \mathcal{V}$, then we obtain $\lambda_1^* > 30$, which is not socially acceptable. This shows the proposed numerical algorithm works well in practice.

VII. CONCLUSIONS AND FUTURE WORKS

In this paper, we have studied the issue of social shaping for dynamic MAS over finite horizons. The system under consideration was a self-sustained dynamic MAS with distributed resource allocations operating at a competitive equilibrium. We presented a conceptual scheme which shows how the social shaping problem is solvable implicitly under some convexity assumptions. As a typical case in the literature, we examined quadratic MAS. Dealing with an LQR problem using quadratic programming and dynamic programming, we proposed two sets of quadratic utility functions under which the resource pricing at the competitive equilibrium is guaranteed to be socially acceptable, i.e., below a prescribed threshold. Finally, we presented a numerical algorithm which provides more accurate bounds on the agents' preferences compared to the proposed analytical sets. As future work, it is suggested to extend the results to the infinite horizon case and consider network constraints in the framework.

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