Social Shaping of Competitive Equilibriums for Resilient Multi-Agent Systems

Yijun Chen, Razibul Islam, Elizabeth Ratnam, Ian R. Petersen, and Guodong Shi

Abstract— **In this paper, we study entirely self-sustained multi-agent systems with decentralized resource allocation. Agents make local resource decisions, and sometimes, trading decisions to maximize their individual payoffs accruing from the utility of consumption and the income or expenditure from trading. A competitive equilibrium is achieved if all agents maximize their individual payoffs; a social welfare equilibrium is achieved if the total agent utilities are maximized. First, we consider multi-agent systems with static local allocation, and prove from duality theory that under general convexity assumptions, the competitive equilibrium and the social welfare equilibrium exist and agree. Next, we define a social shaping problem for a competitive equilibrium under which the optimal resource price is socially acceptable, and show that agent utility functions can be prescribed in a family of socially admissible quadratic functions, under which the pricing at the competitive equilibrium is always below a threshold. Finally, we extend the study to dynamical multi-agent systems where agents are associated with dynamical states from linear processes, and prove that the dynamic competitive equilibrium and social welfare equilibrium continue to exist and coincide with each other.**

I. Introduction

Next generation technologies are leveraging the internet of things (IoT) to support critical infrastructure systems including energy distribution and automotive transportation, and are being organized as interconnected multi-agent systems [1]. Such systems involve data collection, resource allocation, and control coordination between geographically distributed subsystems. Each subsystem, termed an 'agent', is an intelligent functioning unit with its own decision, objective and preference, and remarkably, network-level goals such as consensus, formation, and optimality can be achieved by agents interacting with others over a *network* [2]–[6].

One important problem for multi-agent system operation is efficient resource allocation, where demand and supply must be balanced for efficient and secure operations at the system level. In light of classical welfare economics theory [7], [8], careful pricing of the transmission flow potentially balances the demand against the supply across the entire system. Agents decide on the resource consumed, and perhaps further the resource traded, to maximize their payoffs accruing from the utility of consumption and the income or expenditure from trading. When network supply and demand is balanced, a competitive equilibrium is achieved if all agents maximize their individual payoffs; a social welfare equilibrium is achieved if the total agent utilities are maximized [9].

The concept of resource allocation via a competitive equilibrium has been widely applied in the smart grids and climate-economy systems. In smart grids, agents represent households, and by optimally pricing energy, we ensure the payoffs for all households are maximized subject to the balance of energy supply and demand [10]–[19]. In climate-economy frameworks, agents represent countries, and the optimal price of carbon emissions is calculated under the competitive equilibrium of the carbon market, becoming a benchmark for the social cost of carbon [20]–[24]. However, in both cases, the resilience of the pricing mechanism is potentially a serious challenge even for theoretically optimal equilibrium conditions. For example, residential customers in Texas had to pay electricity bills exceeding previous invoices by a factor of over one hundred during the power shortage event in February 2021 [25]. Moreover, the Paris Agreement was only achieved in 2016 after a decade of negotiations, as the estimated social cost of carbon was perceived as unfair among different nations [26]. Due to the common failure of guaranteeing the smoothness and boundedness of the resource prices in multi-agent systems, it follows that the competitive equilibriums are in need of social shaping at the network level.

In this paper, multi-agent systems with decentralized resource allocation are entirely self-sustained. We refer to "resilient multi-agent systems" in the context of multi-agent systems that can achieve a competitive equilibrium under which the optimal resource price is acceptable for all agents; i.e., below a threshold. In what follows, we first consider multi-agent systems with static local allocation, and we prove that under general convexity assumptions, the competitive equilibrium and the social welfare equilibrium exist and agree. As opposed to the commonly applied KKT arguments [27], our proof is based on duality theory providing a more direct way of connecting the competitive equilibrium to the social welfare equilibrium, and supporting a more general (e.g. nonsmooth) class of utility functions. Next, we propose a social shaping problem for a competitive equilibrium aiming to bound the optimal resource price below a socially acceptable threshold. By means of constructive analysis, we show that agent utility function is prescribed by a family of socially admissible quadratic functions, under which the pricing at the competitive equilibrium is always below a threshold. Finally, we extend the study to dynamical multi-agent systems where agents are associated with dynamical

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states from linear processes, and prove that the dynamic competitive equilibrium and social welfare equilibrium continue to exist and coincide in an optimal control context.

The remainder of the paper is organized as follows. In Section II, we introduce the multi-agent system with static decisions. In Section III, we formulate social shaping of competitive equilibriums for resilient multi-agent systems. In Section IV, we formulate dynamic pricing for resource allocation of multi-agent systems with an underlying dynamical process. Numerical examples are presented in Section V and concluding remarks are presented in Section VI.

II. Static Multi-Agent Systems

A. Competitive Equilibrium for Static Multi-Agent Systems

We consider a multi-agent system (MAS) with n agents. The agents are indexed by $\mathrm{V} = \{1, \dots, n\}$. We consider a basic MAS setup with static decisions on load allocation.

MAS with Static Load Decisions (MAS-SLD). Each agent i holds a local resource of a_i units, and make a (static) decision to allocate $x_i \in \mathbb{R}^{\geq 0}$ units of load for itself. The utility function related to agent i allocating x_i amount of load is $f_i(x_i) : \mathbb{R}^{\geq 0} \mapsto \mathbb{R}$. Consequently, agent i would incur an $a_i - x_i$ amount of surplus ($a_i > x_i$), or a shortcoming ($a_i < x_i$). We assume that there is a connected network among the n agents so that they can balance the surplus and shortcomings through a pricing mechanism. To be precise, each unit of resource across the network is priced at $\lambda \in \mathbb{R}$. Therefore, agent i will yield $(a_i - x_i)\lambda$ in income or spending.

Denoting $\mathbf{x} = (x_1 \dots x_n)^\top$ as the network resource allocation profile, we introduce the following definitions.

Definition 1. *A pair of price-allocation decisions (λ^*, $\mathbf{x}^*$) is a competitive equilibrium for the MAS-SLD if the following conditions hold:*

(i) Each agent i maximizes her combined payoff at x_i^*, i.e., x_i^* is an optimizer of the solution to the following constrained optimization problem:

$$\begin{aligned} \max_{x_i} \quad & f_i(x_i) + \lambda^*(a_i - x_i) \\ \text{s.t.} \quad & x_i \in \mathbb{R}^{\geq 0}. \end{aligned} \tag{1}$$

(ii) The total demand and supply are balanced across the network:

$$\sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} a_i. \tag{2}$$

Definition 2. *A resource allocation profile $\mathbf{x}^\star$ is a social welfare equilibrium for the MAS-SLD if it is a solution to the following optimization problem:*

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i=1}^{n} f_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i, \\ & x_i \in \mathbb{R}^{\geq 0};\ i \in \mathrm{V}. \end{aligned} \tag{3}$$

We present the following result which establishes the equivalence between a competitive equilibrium and a social welfare equilibrium. The result is based only on a concavity assumption for the utility functions f_i.

Theorem 1. *Consider the MAS-SLD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\geq 0}$. Then the social welfare equilibrium(s) and the competitive equilibrium(s) coincide. To be precise, the following statements hold.*

(i) If (λ^*, $\mathbf{x}^*$) is a competitive equilibrium, then $\mathbf{x}^*$ is a social welfare equilibrium.

(ii) If $\mathbf{x}^\star$ is a social welfare equilibrium, then there exists $\lambda^* \in \mathbb{R}$ such that (λ^*, $\mathbf{x}^\star$) is a competitive equilibrium.

Proof. A proof is in [28, Theorem 1]. A numerical example for validating Theorem 1 can be referred to [28, Example 1(i)]. □

Clearly, in this basic multi-agent system setup, the price λ^* associated with a competitive equilibrium could take negative values. From an economic point of view, the resource at every agent must either be consumed or traded, and in cases of an oversupply of resource a negative price for load balancing would occur. From an optimization point of view, the price λ^* is the Lagrange multiplier associated with an equality constraint for a constrained optimization problem, which can take positive or negative sign. The following result indicates that as long as one agent is associated with a non-decreasing utility function, oversupply will not happen.

Proposition 1. *Consider the MAS-SLD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\geq 0}$. Let (λ^*, $\mathbf{x}^*$) be a competitive equilibrium. Then $\lambda^* \geq 0$ if there exists at least one agent $m \in \mathrm{V}$ such that $f_m(\cdot)$ is non-decreasing.*

Proof. See [28, Proposition 1]. □

B. MAS with Trading Decisions

In our standing multi-agent system model, agents only decide on their allocated load x_i, and the surplus/shortcoming $a_i - x_i$ have to go to the network. Now we relax this restriction, and introduce the following extended MAS.

MAS with Static Load and Trading Decisions (MAS-SLTD) On top of the MAS-SLD, each agent i further makes a decision on the traded amount of resource, denoted e_i. This e_i is physically constrained by x_i and a_i in the following way

(i) if $x_i < a_i$, then agent i can sell, in which case $e_i \geq 0$ and $e_i \leq a_i - x_i$;
(ii) if $x_i \geq a_i$, then agent i can only buy, in which case $e_i \leq 0$ and $e_i = a_i - x_i$.

Let λ^* continue to be the price for a unit of shared resource. Denote $\mathbf{e} = (e_1 \dots e_n)^\top$ as the vector representing the traded resource profile across the network.

Definition 3. *A triplet of price-allocation-trade profile (λ^*, $\mathbf{x}^*$, $\mathbf{e}^*$) is a competitive equilibrium for the MAS-SLTD if the following conditions hold:*

(i) Each agent i maximizes her combined payoff at ($\mathbf{x}^*$, $\mathbf{e}^*$) while meeting the physical constraint, i.e., (x_i^*, e_i^*) is

an optimizer of the solution to the following constrained optimization problem:

$$\begin{aligned} \max_{x_i, e_i} \quad & f_i(x_i) + \lambda^* e_i \\ \text{s.t.} \quad & x_i + e_i \le a_i, \\ & x_i \in \mathbb{R}^{\ge 0}, e_i \in \mathbb{R}. \end{aligned} \tag{4}$$

(ii) The total demand and supply are balanced across the network:

$$\sum_{i=1}^{n} e_i^* = 0. \tag{5}$$

Definition 4. *A pair of resource allocation-trade profile $(\mathbf{x}^\star, \mathbf{e}^\star)$ is a social welfare equilibrium for the MAS-SLTD if it is an optimizer of the solution to the following optimization problem:*

$$\max_{\mathbf{x}, \mathbf{e}} \quad \sum_{i=1}^{n} f_i(x_i) \tag{6}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} e_i = 0, \tag{7}$$

$$x_i + e_i \le a_i; i \in \mathrm{V}, \tag{8}$$

$$x_i \in \mathbb{R}^{\ge 0}, e_i \in \mathbb{R}; i \in \mathrm{V}. \tag{9}$$

Theorem 2. *Consider the MAS-SLTD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\ge 0}$. Then the social welfare equilibrium(s) and the competitive equilibrium(s) continue to coincide under the shared load decisions for the agents. To be precise, the following statements hold.*

(i) If $(\lambda^*, \mathbf{x}^*, \mathbf{e}^*)$ is a competitive equilibrium, then $(\mathbf{x}^*, \mathbf{e}^*)$ is a social welfare equilibrium.

(ii) If $(\mathbf{x}^\star, \mathbf{e}^\star)$ is a social welfare equilibrium, then there exists $\lambda^* \in \mathbb{R}$ such that $(\lambda^*, \mathbf{x}^\star, \mathbf{e}^\star)$ is a competitive equilibrium.

Proof. A proof is in [28, Theorem 2]. A numerical example for validating Theorem 2 can be referred to [28, Example 1(ii)]. □

Proposition 2. *Consider the MAS-SLTD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\ge 0}$. Let $(\lambda^*, \mathbf{x}^*, \mathbf{e}^*)$ be a competitive equilibrium under the agent trading decisions. Then there always holds that $\lambda^* \ge 0$.*

Proof. See [28, Proposition 2]. □

III. Social Shaping for Competitive Equilibrium

Consistent with classical welfare economics theory, a competitive equilibrium, despite being a social welfare equilibrium as well, indicates nothing about fairness or sustainability [29]. If the optimal pricing λ^* is too high, agents would quit the system, instead of participating in the self-sustained multi-agent system. When members leave the system, the achievable payoff for the remaining agents would go down. Therefore, the agents share a social responsibility in shaping their utility functions so that λ^* is within a socially acceptable range.

A. Shaping the Competitive Equilibrium

Now we present an approach to achieve a socially acceptable competitive equilibrium, by synthesizing a class of utility functions from which agents can select. We make the following assumption.

Assumption 1. Each f_i is represented by $f_i(x_i) = -\frac{1}{2}b_i x_i^2 + k_i x_i$, where $b_i \in \mathbb{R}^{>0}$ and $k_i \in \mathbb{R}^{\ge 0}$; a utility function f_i is socially admissible if there hold $k_i \in [k_{\min}, k_{\max}]$ and $b_i \in [b_{\min}, b_{\max}]$.

Let $\lambda^\dagger > 0$ represent the highest pricing for λ^* that agents can accept, and we term such a competitive equilibrium $\lambda^* \le \lambda^\dagger$ a *socially resilient equilibrium*. Let $\mathbf{a} = (a_1 \dots a_n)^\top$ represent the network resource allocation profile, and let $C := \sum_{i=1}^{n} a_i$ represent the network resource capacity. Assuming C and $\mathbf{a}$ are given network characteristics, we consider the following problem of shaping the competitive equilibrium.

Problem. (Social Competitive Equilibrium Shaping) Consider the MAS-SLD. Find the range for $k_{\min}$, $k_{\max}$, $b_{\min}$, $b_{\max}$ under which there always exists a competitive equilibrium that leads to $0 \le \lambda^* \le \lambda^\dagger$, for all socially admissible utility functions.

B. Socially Admissible Utility Functions

Denote $\mathbf{k} = (k_1, \dots, k_n)^\top$ and $\mathbf{b} = (b_1, \dots, b_n)^\top$. For two vectors $\mathbf{l} = (l_1, \dots, l_n)^\top$ and $\mathbf{l}' = (l'_1, \dots, l'_n)^\top$, we write $\mathbf{l} \preceq \mathbf{l}'$ if there holds $l_i \le l'_i$ for all $i \in \mathrm{V}$. In other words, $\preceq$ defines a partial order for all vectors in $\mathbb{R}^n$.

Define

$$\begin{aligned} \mathscr{S}_* := \Big\{ \big(k_{\min}, k_{\max}, b_{\min}, b_{\max}\big) \in \mathbb{R}^4_{\ge 0} : \frac{n k_{\min}}{b_{\max}} \ge C; \\ -\frac{n k_{\min}}{b_{\max}} + \frac{n k_{\max}}{b_{\min}} \le C; -\frac{n \lambda^\dagger}{b_{\max}} + \frac{n k_{\max}}{b_{\min}} \le C \Big\}. \end{aligned} \tag{10}$$

We present the following theorem.

Theorem 3. *Consider the MAS-SLD. Let Assumption 1 hold. The following statements hold.*

(i) The competitive equilibrium is unique, and therefore, there exists a well-defined mapping, denoted by $\mathcal{F}(\cdot, \cdot)$, that maps $(\mathbf{k}, \mathbf{b})$ to $\lambda^* := \mathcal{F}(\mathbf{k}, \mathbf{b})$ where λ^* belongs to the competitive equilibrium.

(ii) The competitive equilibrium is always socially resilient (ie $\lambda^* \le \lambda^\dagger$) for all socially admissible utility functions as long as $\big(k_{\min}, k_{\max}, b_{\min}, b_{\max}\big) \in \mathscr{S}_*$.

(iii) Let $\big(k_{\min}, k_{\max}, b_{\min}, b_{\max}\big) \in \mathscr{S}_*$ be given. The mapping $\mathcal{F}(\cdot, \cdot)$ is monotone under the partial order $\preceq$ over $\mathbf{k}$ in the sense that

$$\mathcal{F}(\mathbf{k}, \mathbf{b}) \le \mathcal{F}(\mathbf{k}', \mathbf{b})$$

for all socially admissible $\mathbf{k} \preceq \mathbf{k}'$.

Proof. See [28, Theorem 3]. □

IV. Dynamic Multi-Agent Systems

A. MAS with Dynamic Agent Load/Trading Decisions

Here we consider the load balancing problem for dynamical multi-agent systems.

MAS with Dynamic Load/Trading Decisions (MAS-DLTD). Each agent $i \in \mathrm{V}$ is associated with a dynamical state $\mathbf{y}_i(t) \in \mathbb{R}^m$, described by

$$\mathbf{y}_i(t+1) = \mathbf{A}_i \mathbf{y}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \quad t \in \mathcal{T}, \tag{11}$$

where $\mathbf{u}_i(t) \in \mathbb{R}^d$ is the control input, and $\mathbf{A}_i$ and $\mathbf{B}_i$ are real matrices with proper dimensions. The time steps are indexed by $\mathcal{T} = \{0, \dots, T-1\}$. Associated with $t \in \mathcal{T}$, agent i incurs a utility function $f_i(\mathbf{y}_i(t), \mathbf{u}_i(t))$; the terminal utility for agent i is $\Phi_i(\mathbf{y}_i(T))$. Upon taking the control action $\mathbf{u}_i(t)$, the required resource is defined by the function $h_i(\mathbf{u}_i(t))$. If we, on a winter's day, turn on more heat in a room, the temperature will start rising. The change is largest in the beginning, and "eventually" the temperature will approach a new steady state value. Each agent can produce an $a_i(t)$ units of resource at time t, and also makes a trading decision $e_i(t)$ units of resource over the network at time t. Similarly,

(i) if $h_i(\mathbf{u}_i(t)) < a_i(t)$, then agent i can sell, in which case $e_i(t) \geq 0$ and $e_i(t) \leq a_i(t) - h_i(\mathbf{u}_i(t))$;

(ii) if $h_i(\mathbf{u}_i(t)) \geq a_i(t)$, then agent i will buy, in which case $e_i(t) \leq 0$ and $e_i(t) = a_i(t) - h_i(\mathbf{u}_i(t))$.

We denote $\boldsymbol{\lambda} = (\lambda_0 \dots \lambda_{T-1})^\top$ as the pricing vector through the time horizon, where λ_t is the unit price for traded energy at step t. Consequently, the payoff of agent i throughout $[0, T]$ is described by

$$\sum_{t=0}^{T-1} \Big(f_i(\mathbf{y}_i(t), \mathbf{u}_i(t)) + \lambda_t e_i(t) \Big) + \Phi_i(\mathbf{y}_i(T)).$$

Denote $\mathbf{y}(t) = (\mathbf{y}_1(t)^\top \dots \mathbf{y}_n(t)^\top)^\top$, $\mathbf{u}(t) = (\mathbf{u}_1(t)^\top \dots \mathbf{u}_n(t)^\top)^\top$, and $\mathbf{e}(t) = (\mathbf{e}_1(t)^\top \dots \mathbf{e}_n(t)^\top)^\top$. Further define $\mathbf{Y} = (\mathbf{y}(0)^\top \dots \mathbf{y}(T)^\top)^\top$, $\mathbf{U} = (\mathbf{u}(0)^\top \dots \mathbf{u}(T-1)^\top)^\top$ and $\mathbf{E} = (\mathbf{e}(0)^\top \dots \mathbf{e}(T-1)^\top)^\top$. Also introduce $\mathbf{U}_i = (\mathbf{u}_i(0)^\top \dots \mathbf{u}_i(T-1)^\top)^\top$, $\mathbf{E}_i = (\mathbf{e}_i(0)^\top \dots \mathbf{e}_i(T-1)^\top)^\top$ and $\mathbf{a}_i = (a_i(0)^\top \dots a_i(T-1)^\top)^\top$.

Definition 5. *Let $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^{mn}$ be given. A triple of price-control-trading profiles ($\boldsymbol{\lambda}^*, \mathbf{U}^*, \mathbf{E}^*$) is a dynamic competitive equilibrium if the following conditions hold:*

(i) Each agent i maximizes its combined payoff under $\mathbf{U}_i^*$ and $\mathbf{E}_i^*$:

$$\begin{aligned} \max_{\mathbf{U}_i, \mathbf{E}_i} \quad & \sum_{t=0}^{T-1} \Big(f_i(\mathbf{y}_i(t), \mathbf{u}_i(t)) + \lambda_t^* e_i(t) \Big) + \Phi_i(\mathbf{y}_i(T)) \\ \text{s.t.} \quad & \mathbf{y}_i(t+1) = \mathbf{A}_i \mathbf{y}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \quad t \in \mathcal{T}, \\ & e_i(t) \leq a_i(t) - h_i(\mathbf{u}_i(t)), \quad t \in \mathcal{T}; \end{aligned} \tag{12}$$

(ii) The total demand and supply are balanced across the network for all time, i.e., there holds

$$\sum_{i=1}^{n} e_i(t) = 0, \quad t \in \mathcal{T}. \tag{13}$$

Definition 6. *Let $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^{mn}$ be given. A pair of control-trading profiles $(\mathbf{U}^\star, \mathbf{E}^\star)$ is a dynamic social welfare equilibrium if it is a solution to the following optimal control problem:*

$$\max_{\mathbf{U}, \mathbf{E}} \quad \sum_{i=1}^{n} \Big(\sum_{t=0}^{T-1} f_i(\mathbf{y}_i(t), \mathbf{u}_i(t)) + \Phi_i(\mathbf{y}_i(T)) \Big) \tag{14}$$

$$\text{s.t.} \quad \mathbf{y}_i(t+1) = \mathbf{A}_i \mathbf{y}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \quad t \in \mathcal{T}, \ i \in \mathrm{V}, \tag{15}$$

$$e_i(t) \leq a_i(t) - h_i(\mathbf{u}_i(t)), \quad t \in \mathcal{T}, \ i \in \mathrm{V}, \tag{16}$$

$$\sum_{i=1}^{n} e_i(t) = 0, \quad t \in \mathcal{T}. \tag{17}$$

B. Dynamic Competitive Equilibrium

We impose the following assumption.

Assumption 2. (i) the Φ_i are concave functions for $i \in \mathrm{V}$; (ii) the f_i are concave functions for $i \in \mathrm{V}$; (iii) the h_i are nonnegative convex functions for $i \in \mathrm{V}$, and $h_i(\mathbf{z}) < b$ defines a bounded open set of $\mathbf{z}$ in $\mathbb{R}^m$ for $b > 0$; (iv) $\sum_{i=1}^{n} a_i(t) > 0$ for all $t \in \mathcal{T}$.

We present the following result which establishes the similar connection between the competitive equilibrium and the social welfare equilibrium under this dynamic setting.

Theorem 4. *Consider the MAS-DLTD with $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^{mn}$ be given. Let Assumption 2 hold. The dynamic social welfare equilibrium(s) and the dynamic competitive equilibrium(s) coincide and the following statements hold.*

(i) If $(\boldsymbol{\lambda}^*, \mathbf{U}^*, \mathbf{E}^*)$ is a dynamic competitive equilibrium, then $(\mathbf{U}^*, \mathbf{E}^*)$ is a dynamic social welfare equilibrium.

(ii) If $(\mathbf{U}^\star, \mathbf{E}^\star)$ is a dynamic social welfare equilibrium, then there exists $\boldsymbol{\lambda}^* \in \mathbb{R}^T$ such that $(\boldsymbol{\lambda}^*, \mathbf{U}^\star, \mathbf{E}^\star)$ is a competitive equilibrium.

Proof. See [28, Theorem 4]. □

V. Numerical Examples

A. MAS with Static Decisions

Example 1. Consider a multi-agent system with four agents. The utility function for agent i is in the quadratic form $f_i = -\frac{1}{2} b_i x_i^2 + k_i x_i$ for $i = 1, 2, 3, 4$. We consider two pairs of system parameters

$$\mathbf{b} = (2, 5, 3, 4)^\top \quad \mathbf{k} = (21, 17, 23, 13)^\top; \tag{PM.1}$$

$$\mathbf{b}' = (2, 5, 3, 4)^\top \quad \mathbf{k}' = (25, 22, 24, 14)^\top. \tag{PM.2}$$

Let the network resource capacity $C = \sum_{i=1}^{4} a_i$ take values in an interval $(0, 40)$. We sample the interval $(0, 40)$ uniformly with a step-size 0.8 to obtain 50 different values for C. For each C, we compute the optimal prices of the system under MAS-SLD and MAS-SLTD.

For MAS-SLD, the optimal dual variables $\lambda_{\mathrm{SLD}}^{*(\mathrm{PM.1})}$ and $\lambda_{\mathrm{SLD}}^{*(\mathrm{PM.2})}$ are computed for 50 times corresponding to each value of C by solving (3), respectively, under the parameter settings (PM.1) and (PM.2). For MAS-SLTD, the optimal dual variables $\tau_{\mathrm{SLTD}}^{*(\mathrm{PM.1})}$ and $\tau_{\mathrm{SLTD}}^{*(\mathrm{PM.2})}$ related to the equality constraint (7) are also computed for 50 times corresponding to each value of C by solving (6)-(9), respectively, under the parameter setting (PM.1) and (PM.2), and then we take $\lambda^{*(\mathrm{PM.1})}_{\mathrm{SLTD}} = -\tau^{*(\mathrm{PM.1})}_{\mathrm{SLTD}}$ and $\lambda^{*(\mathrm{PM.2})}_{\mathrm{SLTD}} = -\tau^{*(\mathrm{PM.2})}_{\mathrm{SLTD}}$. In Fig. 1, we plot 50 points of optimal prices versus C, to obtain an approximate trajectory of optimal price as a function of C.

From Fig. 1 we observe that the optimal price λ^*_{SLD} in MAS-SLD can indeed take negative values; while the optimal price $\lambda^*_{\mathrm{SLTD}}$ in MAS-SLTD is always non-negative. These observations are consistent with Proposition 1 and Proposition 2. Moreover, for both MAS-SLD and MAS-SLTD, we observe in Fig. 1 that the optimal prices $\lambda^*_{\mathrm{SLD}}, \lambda^*_{\mathrm{SLTD}}$ are decreasing as the network resource capacity C increases. □

Example 2. Consider a MAS-SLD with three agents and network capacity $C = 18$. Each agent's utility function is set as the quadratic form $f_i = -\frac{1}{2} b_i x_i^2 + k_i x_i$ for $i = 1, 2, 3$. The system's highest pricing for λ^* that agents can accept socially is assumed to be $\lambda^\dagger = 39$. Take $b_{\min} = 4$, $b_{\max} = 6$, $k_{\min} = 40$, and $k_{\max} = 50$. We can verify such a configuration of $(b_{\min}, b_{\max}, k_{\min}, k_{\max})$ is a point in $\mathscr{S}_*$ defined in (10).

(i) Let $\mathbf{b}$ be fixed to be $\mathbf{b} = (4, 5, 6)^\top$. Take $k_3 = 44, 48$, respectively. We sample the space for $(k_1, k_2) \in [40, 50]^2$ and compute the optimal pricing λ^* by solving the optimal dual variable of (3). Then we plot the contour maps for the optimal price as a function of k_1 and k_2 in the first row of Fig. 2.

(ii) Let $\mathbf{k}$ be fixed to be $\mathbf{k} = (44, 46, 48)^\top$. Take $b_3 = 4.8, 5.2$, respectively. We sample the space for $(b_1, b_2) \in [4, 6]^2$ and compute the optimal pricing λ^* by solving the optimal dual variable of (3). Then we plot the contour maps for the optimal price as a function of b_1 and b_2 in the second row of Fig. 2.

In Fig. 2 we observe that the maximum value for the price λ^* is 20, which is lower than $\lambda^\dagger = 39$. This illustrates all socially admissible utility functions for parameters in the set $\mathscr{S}_*$ lead to socially acceptable prices, providing a validation for Theorem 3(ii). From the first row of Fig. 2, the optimal price is monotone under the partial order $\preceq$ with respect to $\mathbf{k}$, which is consistent with Theorem 3(iii). □

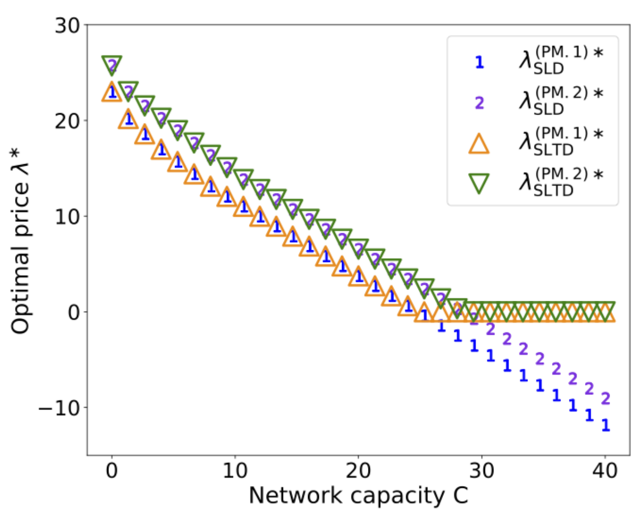

Fig. 1: The optimal prices as functions of the network resource capacity for MAS-SLD and MAS-SLTD under two different parameter settings (PM.1) and (PM.2).

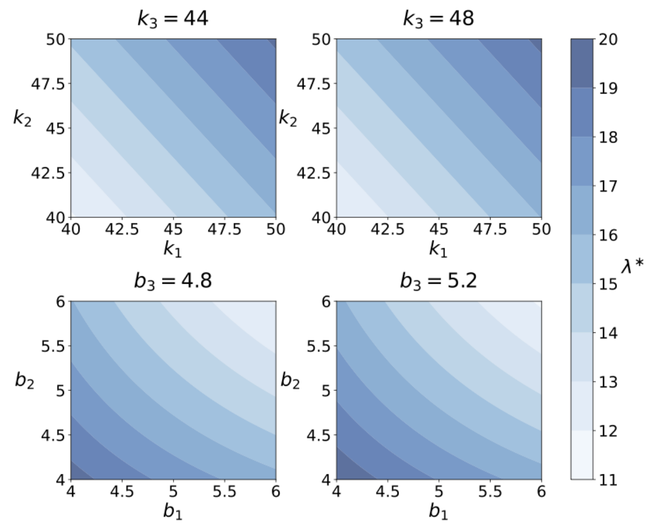

Fig. 2: The first row: contour maps for the optimal price as a function of k_1 and k_2. The second row: contour maps for the optimal price as a function of b_1 and b_2.

B. MAS with Dynamic Decisions

Example 3. Consider a MAS-DLTD with three agents. The setting for this MAS-DLTD can be referred to [28, Example 4]. Let the time horizon take the value of $T = 30$. We compute the dynamic social welfare equilibrium $(\mathbf{U}^\star, \mathbf{E}^\star)$ by solving the optimization problem (14)-(17) and the optimal dual variables $-\boldsymbol{\lambda}^*$ corresponding to (17). Given $\boldsymbol{\lambda}^*$, we further compute the dynamic competitive equilibrium $(\mathbf{U}^*, \mathbf{E}^*)$ by solving (12). In Fig. 3, we plot the dynamic social welfare equilibrium and the dynamic competitive equilibrium. In Fig. 3, we observe that the dynamic social welfare equilibrium and the dynamic competitive equilibrium agree, which is consistent with Theorem 4. □

VI. Conclusions

In this paper, we studied multi-agent systems with decentralized resource allocation without external resource supply. For multi-agent systems with static local allocation, we showed that under general convexity assumptions, the competitive equilibrium and the social welfare equilibrium exist and agree using a duality analysis. An important aspect of this paper is our formulation of the social shaping problem for competitive equilibriums, where the optimal pricing is associated with an upper bound. We also presented an explicit family of socially admissible utility functions under which the optimal pricing at a competitive equilibrium is always socially acceptable. A major contribution of this paper is our study of dynamical multi-agent systems and we have generalized it in an optimal control context. We proved

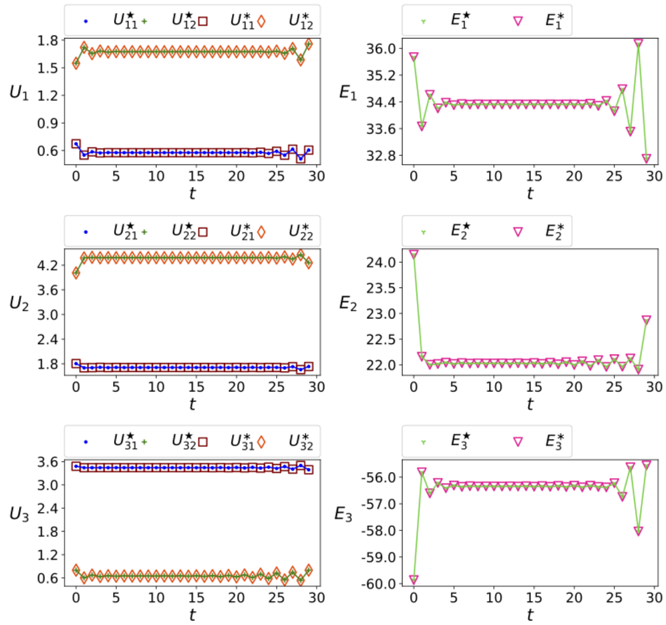

Fig. 3: The dynamic social welfare equilibrium (with superscript $\bigstar$) and competitive equilibrium (with superscript $*$).

that the dynamic competitive equilibrium and social welfare equilibrium continue to exist and coincide with each other. Future research to smooth dynamic optimal pricing for MAS with dynamic decisions is possible. Future work to construct a range of socially admissible utility functions in a generic way is also possible.

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