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Competitive equilibriums and social shaping for multi-agent systems[☆]

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ABSTRACT

In this paper, we study multi-agent systems with decentralized resource allocations. Agents have local demand and resource supply, and are interconnected through a network designed to support sharing of the local resource; and the network has no external resource supply. It is known from classical welfare economics theory that by pricing the flow of resource, balance between the demand and supply is possible. Agents decide on the consumed resource, and perhaps further the traded resource as well, to maximize their payoffs considering both the utility of the consumption, and the income from the trading. When the network supply and demand are balanced, a competitive equilibrium is achieved if all agents maximize their individual payoffs, and a social welfare equilibrium is achieved if the total agent utilities are maximized. First, we consider multi-agent systems with static local allocations, and prove from duality theory that under general concavity assumptions, the competitive equilibrium and the social welfare equilibrium exist and agree. Next, we show that the agent utility functions can be prescribed in a family of socially admissible functions, under which the resource price at the competitive equilibrium is kept below a threshold. Finally, we extend the study to dynamical multi-agent systems where agents are associated with dynamical states from linear processes, and prove that the dynamic competitive equilibrium and the dynamic social welfare equilibrium continue to exist and coincide with each other. In addition, we also present a recursive representation of the competitive equilibriums using dynamic programming, and a receding horizon approach for smoothing the dynamic pricing as a dynamic competitive equilibrium social shaping method.

1. Introduction

Next generation technologies are leveraging the internet of things (IoT) to support critical infrastructure systems including energy distribution and automotive transportation, and are being organized as interconnected multi-agent systems (Mesbahi & Egerstedt, 2010). Such systems involve data collection, resource allocation, and control coordination between geographically distributed subsystems. Each subsystem, termed an 'agent', is an intelligent functioning unit with its own decisions, objectives and preferences, and remarkably, network-level goals such as consensus, formation, and optimality can be achieved by agents interacting with others over a *network* (Jadbabaie, Lin, & Morse, 2003; Martinez, Cortes, & Bullo, 2007; Nedić, Ozdaglar, & Parrilo, 2010; Olfati-Saber & Murray, 2004; Tsitsiklis, 1984). The underlying network for multi-agent systems can be physical such as transmission lines in a power grid, non-physical such as wireless communication channels, or a combination of the two. The key promise of organizing subsystems into networked multi-agent systems is a radical improvement in scalability, efficiency, and sustainability through shared inputs and outputs, and coordinated decisions and controls.

One important problem for multi-agent system operation is efficient resource allocation, where demand and supply must be balanced for efficient and secure operations at the system level. In a typical resource allocation problem, agents have local demand and internal and external resource suppliers, interconnected through a network that allows for transmission of the resource. In light of classical welfare economics theory (Mas-Colell, Whinston, Green, et al., 1995), careful pricing of the transmission flow potentially balances the demand against the supply across the entire system. Agents decide on the resource consumed, and perhaps further the resource traded, to maximize their payoffs considering both the utility from consumption, and income from the trading. When network supply and demand is balanced, a competitive equilibrium is achieved if all agents maximize their

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individual payoffs; a social welfare equilibrium is achieved if the total agent utilities are maximized, which in turn maximizes the overall system-level payoff.

The concept of operating a multi-agent system as a market via optimal pricing under a competitive equilibrium has applications in smart grid operations and climate-economy systems. In smart grids, agents represent households, and by optimally pricing energy, we ensure the payoff for all households are maximized subject to the balance of energy supply and demand (Alvarado, Meng, DeMarco, & Mota, 2001; Chen, Li, Low, & Doyle, 2010; Jadhav, Patne, & Guerrero, 2018; Jokić, Lazar, & Van den Bosch, 2010; Knudsen, Hansen, & Annaswamy, 2015; Li, Chen, & Low, 2011; Li, Lian, Conejo, & Zhang, 2020; Muthirayan, Kalathil, Poolla, & Varaiya, 2019; Papadaskalopoulos & Strbac, 2013; Singh, Kumar, & Xie, 2018; Stegink, De Persis, & Van der Schaft, 2016; Zhang & Papachristodoulou, 2015). In climate-economy systems, agents represent countries, and optimal pricing of carbon emissions ensures the interests of each country are met subject to a carbon emission supply–demand balance (Nordhaus, 2011, 2017; Nordhaus & Boyer, 2003). However, in both cases, the optimally computed price is potentially not socially acceptable. For example, in February 2021, the electricity price in Texas went to an unacceptably high rate after widespread power outages. Consumers who were involved in the market-based contract must pay sky-high electricity bills, which exceeded their budgets (Blumsack, 2022; Morehouse, 2022). Moreover, the carbon emissions trading scheme under the Kyoto Protocol was widely criticized by researchers, as the estimated social cost of carbon was deemed as unacceptable among different regions (Kuyper, Schroeder, & Linnér, 2018; Napoli, 2012). We refer to Kellett, Weller, Faulwasser, Grüne, and Semmler (2019) for an excellent introduction to the dynamic integration of climate and economy models from a feedback system perspective. Both examples raise the problem that boundedness of the resource price is not guaranteed.

However, in the extensive literature of economics and engineering, early and recent studies focus on the rapid changes in the pricing process termed price volatility, rather than the resource price itself. The Black Scholes formula (Black & Scholes, 1973) and Heston's extension (Heston, 1993) are the most representative models of stochastic price volatility. In Kizilkale and Mannor (2010), Kizilkale and Mannor argue that previous models do not penalize price volatility in the system-level objective. They modify the system objective to account for price volatility and construct a dynamic game-theoretic framework for power markets. In Tsitsiklis and Xu (2015), Tsitsiklis and Xu suggest a different dynamic game-theoretic model of electricity markets with the incorporation of ancillary service cost, and propose a pricing mechanism that has the potential to reduce the peak load. In Wei, Malekian, and Ozdaglar (2014), Wei, Malekian and Ozdaglar consider the dual version of the system-level welfare optimization problem. An explicit penalty term on the $L-2$ norm of price volatility is introduced in the system objective, which allows trade-offs between price volatility and social welfare. These methods strike trade-offs between economic efficiency and price volatility. In other words, one has to compromise system efficiency in order to reduce price volatility, which causes an inevitable supply–demand mismatch. Furthermore, to the authors' best knowledge, none of the existing pricing mechanisms can limit the range of the resource price required at a competitive equilibrium in a formal way. It indicates that the resource price might remain unacceptable among agents who might opt out of the system afterwards. This can be a significant practical problem due to its failure of considering implicit agents' budgetary constraints (Sinha & Anastasopoulos, 2017).

Consequently, it motivates us to focus on the resource price itself and propose a new social shaping problem for a competitive equilibrium aiming to bound the resource price below a socially acceptable threshold. To this end, Section 3 considers parameterized utility functions whose parameters are completely abstracted from agents' preferences and prescribe a range for the parameters of utility functions to ensure that the resource price under a competitive equilibrium is socially acceptable for all agents without mismatching supply and demand. The idea of introducing parameterized utility functions comes from the concept of smart thermostat agent in the AEP Ohio gridSMART Demonstration (Widergren, Fuller, Marinovici and Somani, 2014; Widergren et al., 2014). Agents are provided with an interface (usually a slider) to specify their preference settings for relative comfort and savings. The two sides of the slider represent the maximum comfort and maximum savings, respectively. The parameters in utility functions reflect the extent to which agents have preferences towards comfort or towards savings. When a agent chooses to move the slider towards comfort, it indicates that the agent is willing to maintain comfort regardless of the cost of resources. When a agent decides to move the slider towards savings, it implies that the agent would like to sacrifice comfort for cost savings.

Thermostatically controlled loads (TCLs) in the home, such as air conditioners, heat pumps, water heaters, and refrigerators, are well suited to energy arbitrage. The TCLs' function is to regulate their internal temperature within specific bounds. There is a specific average power that is necessary to fulfil this function under nominal local management. Many TCLs offer two modes of operation: powered at a set level and unpowered, and keep temperatures within dead-bands in both. Agents are satisfied if temperatures remain within their designated settings. Prior studies have showed the possibility to model an aggregation of heterogeneous TCLs by dynamical systems (Hao, Sanandaji, Poolla, & Vincent, 2013; Mathieu, Kamgarpour, Lygeros, Andersson, & Callaway, 2014; Mortensen & Haggerty, 1990). It motivates us to extend our study to dynamical multi-agent systems in Section 4.

Contribution. In this paper, multi-agent systems with decentralized resource allocation are entirely selfsustained. Our contributions are summarized as follows:

(i) We first consider multi-agent systems with static local allocation, and prove from a duality argument that under general concavity assumptions, the competitive equilibrium and the social welfare equilibrium exist and agree.
(ii) We then formulate a new social shaping problem to investigate the case when the optimal resource price at the competitive equilibrium is associated with an upper bound for social acceptance. We focus on a fundamental class of quadratic utility functions, and show that the social shaping problem can be explicitly solved by prescribing a family of socially admissible quadratic functions that agents can select from.
(iii) We extend the study to dynamical multi-agent systems where agents are associated with dynamical states from linear processes, and prove that the dynamic competitive equilibrium and the dynamic social welfare equilibrium continue to exist and coincide in the context of optimal control, again from a duality perspective. We also present a recursive way of representing and computing the dynamic competitive equilibrium in view of the dynamic programming principle. In order to shape the dynamic pricing in the sense that the pricing trajectory would be stationary, we propose a receding horizon approach for smoothing the dynamic pricing.

Some preliminary results of our work were presented in Chen, Islam, Ratnam, Petersen, and Shi (2021). The remainder of the paper is organized as follows. In Section 2, we introduce the multi-agent system with static decisions. In Section 3, we define a new social shaping problem for multi-agent systems and prescribe a family of socially admissible quadratic functions under which the optimal resource price is always acceptable for all agents. In Section 4, we formulate dynamic pricing for resource allocation of multi-agent systems with an underlying dynamical process. Some concluding remarks are presented in Section 5. Our codes for numerical examples are open-sourced at https://github.com/chyj528/MAS-Social-Shaping.

2. Static multi-agent systems

In this section, we study multi-agent systems with static resource allocation and load decisions.

2.1. Competitive equilibrium for static multi-agent systems

We consider a multi-agent system (MAS) with n agents. The agents are indexed by $\mathrm{V}=\{1,\ldots,n\}$. We consider a basic MAS setup with static agent decisions on load allocations.

MAS with Static Agent Load Decisions (MAS-SALD). Each agent i holds a local resource of a_i units, and makes a (static) decision to allocate $x_i\in\mathbb{R}^{\geq 0}$ units of load for itself. The utility function related to agent $i\in\mathrm{V}$ allocating x_i amount of load is $f_i(x_i):\mathbb{R}^{\geq 0}\mapsto\mathbb{R}$. Consequently, agent i would incur an a_i-x_i amount of surplus ($a_i>x_i$), or a shortcoming ($a_i<x_i$). We assume that there is a connected network among the n agents so that they can balance the surplus and shortcomings through a pricing mechanism. To be precise, each unit of resource moved across the network is priced at $\lambda\in\mathbb{R}$. Therefore, agent i will yield $(a_i-x_i)\lambda$ in income or expenditure.

Denoting $\mathbf{x}=(x_1\ldots x_n)^\top\in(\mathbb{R}^{\geq 0})^n$ as the network resource allocation profile, we introduce the following definitions.

Definition 1. A pair of price-allocation decisions $(\lambda^*,\mathbf{x}^*)$ is a competitive equilibrium for the MAS-SALD if the following conditions hold:

(i) each agent i maximizes her combined payoff at x_i^*, i.e., x_i^* is an optimizer of the solution to the following constrained optimization problem:

$$\begin{aligned}\max_{x_i}\quad & f_i(x_i)+\lambda^*(a_i-x_i)\\ \text{s.t.}\quad & x_i\in\mathbb{R}^{\geq 0}.\end{aligned}\tag{1}$$

(ii) the total demand and supply are balanced across the network:

$$\sum_{i=1}^n x_i^*=\sum_{i=1}^n a_i.\tag{2}$$

Definition 2. A resource allocation profile $\mathbf{x}^\star$ is a social welfare equilibrium for the MAS-SALD if it is a solution to the following optimization problem:

$$\begin{aligned}\max_{\mathbf{x}}\quad & \sum_{i=1}^n f_i(x_i)\\ \text{s.t.}\quad & \sum_{i=1}^n x_i=\sum_{i=1}^n a_i,\\ & x_i\in\mathbb{R}^{\geq 0};\ i\in\mathrm{V}.\end{aligned}\tag{3}$$

We present the following result which establishes the equivalence between a competitive equilibrium and a social welfare equilibrium. The result is based only on a concavity assumption for the utility functions f_i.

Theorem 1. *Consider the MAS-SALD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\geq 0}$. Then the social welfare equilibrium(s) and the competitive equilibrium(s) coincide. To be precise, the following statements hold.*

(i) If $(\lambda^*,\mathbf{x}^*)$ is a competitive equilibrium, then $\mathbf{x}^*$ is a social welfare equilibrium.

(ii) If $\mathbf{x}^\star$ is a social welfare equilibrium, then there exists $\lambda^*\in\mathbb{R}$ such that $(\lambda^*,\mathbf{x}^\star)$ is a competitive equilibrium.

Proof. (i) Let $(\lambda^*,\mathbf{x}^*)$ be a competitive equilibrium. The proof proceeds by contradiction. Suppose that $\mathbf{x}^*$ is not a social welfare equilibrium. Then there must exist $\bar{\mathbf{x}}^*$ such that $\sum_{i=1}^n\bar{x}_i^*=\sum_{i=1}^n x_i^*=\sum_{i=1}^n a_i$, and $\sum_{i=1}^n f_i(x_i^*)<\sum_{i=1}^n f_i(\bar{x}_i^*)$. Consequently, there holds

$$\sum_{i=1}^n\Big(f_i(x_i^*)+\lambda^*(a_i-x_i^*)\Big)<\sum_{i=1}^n\Big(f_i(\bar{x}_i^*)+\lambda^*(a_i-\bar{x}_i^*)\Big).\tag{4}$$

This implies that there is at least one $m\in\mathrm{V}$ such that

$$f_m(x_m^*)+\lambda^*(a_m-x_m^*)<f_m(\bar{x}_m^*)+\lambda^*(a_m-\bar{x}_m^*),$$

which contradicts the fact that $(\lambda^*,\mathbf{x}^*)$ is a competitive equilibrium.

(ii) We propose a proof using duality. To be consistent with the literature on duality theory for continuous optimization, we denote $g_i=-f_i$, and rewrite (3) as

$$\begin{aligned}\min_{\mathbf{x}}\quad & \sum_{i=1}^n g_i(x_i)\\ \text{s.t.}\quad & \sum_{i=1}^n x_i=\sum_{i=1}^n a_i;\ x_i\in\mathbb{R}^{\geq 0},\ i\in\mathrm{V}.\end{aligned}\tag{5}$$

Let $\mathbf{x}^\star$ be a social welfare equilibrium.[1] Then from its definition there holds $\sum_{i=1}^n x_i^\star=\sum_{i=1}^n a_i$. Since (5) is a convex optimization problem with a linear equality constraint, strong duality holds (Boyd, Boyd, & Vandenberghe, 2004) and we denote the optimal primal and dual costs of (5) as p_* and d^*, respectively.

The Lagrange function of (5) is

$$L(\mathbf{x},\lambda)=\sum_{i=1}^n g_i(x_i)+\lambda\Big(\sum_{i=1}^n x_i-\sum_{i=1}^n a_i\Big):(\mathbb{R}^{\geq 0})^n\times\mathbb{R}\mapsto\mathbb{R}.$$

Then we introduce $L^*(\lambda)=\min_{\mathbf{x}\in(\mathbb{R}^{\geq 0})^n}L(\mathbf{x},\lambda)$. If λ^* is dual optimal (i.e., $\lambda^*\in\arg\max_{\lambda\in\mathbb{R}}L^*(\lambda)$), there holds from strong duality (Boyd et al., 2004) that

$$\begin{aligned}d^*=L^*(\lambda^*)&=\min_{\mathbf{x}\in(\mathbb{R}^{\geq 0})^n}L(\mathbf{x},\lambda^*) &(6)\\ &\leq L(\mathbf{x}^\star,\lambda^*) &(7)\\ &=\sum_{i=1}^n g_i(x_i^\star) &(8)\\ &=p^*. &(9)\end{aligned}$$

This implies the inequality from the above equation actually holds at equality:

$$\mathbf{x}^\star\in\arg\min_{\mathbf{x}\in(\mathbb{R}^{\geq 0})^n}L(\mathbf{x},\lambda^*).\tag{10}$$

Note that $L(\mathbf{x},\lambda^*)=\sum_{i=1}^n\big(g_i(x_i)+\lambda^*(x_i-a_i)\big)$ implies

$$x_i^\star\in\arg\max_{x_i\in\mathbb{R}^{\geq 0}}\big(f_i(x_i)+\lambda^*(a_i-x_i)\big).\tag{11}$$

Thus, we have proved that $(\lambda^*,\mathbf{x}_i^\star)$ is a competitive equilibrium. □

[1] Note that $\mathbf{x}^\star$ must be finite as the feasible set of $\mathbf{x}$ is compact.

Clearly, in this basic multi-agent system setup, the price λ^* associated with a competitive equilibrium could take negative values. From an economic point of view, the resource at every agent must either be consumed or traded, and in cases of an oversupply of resource a negative price for load balancing would occur. From an optimization point of view, the price λ^* is the Lagrange multiplier associated with an equality constraint for a constrained optimization problem, which can take positive or negative values. Proposition 1 indicates that as long as one agent is associated with a non-decreasing utility function, oversupply will not happen.

Proposition 1. *Consider the MAS-SALD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\geq 0}$. Let $(\lambda^*, \mathbf{x}^*)$ be a competitive equilibrium. Then $\lambda^* \geq 0$ if there exists at least one agent $m \in \mathrm{V}$ such that $f_m(\cdot)$ is non-decreasing.*

Proof. Let $f_m(\cdot)$ be non-decreasing. Assume $\lambda^* < 0$. Then $f_m(x_m) + \lambda^*(a_m - x_m)$ is a strictly increasing function with respect to x_m. Therefore, there cannot be a finite x_m^* such that $x_m^* \in \arg\max_{x_m \in \mathbb{R}^{\geq 0}} \big(f_m(x_m) + \lambda^*(a_m - x_m)\big)$, contradicting the definition of the competitive equilibrium. □

2.2. MAS with trading decisions

In our standing multi-agent system model, agent i only decides on its allocated load x_i with the surplus/shortcoming $a_i - x_i$ returned to the network. Next we relax the network restriction, and introduce the following extended MAS.

MAS with Static Agent Load and Trading Decisions (MAS-SALTD) Here we extend the MAS-SALD. Each agent i further makes a decision on the traded amount of resource, denoted e_i. As one recent example, e_i is physically constrained by x_i and a_i in the following way:

(i) if $x_i < a_i$, then agent i can sell, in which case $e_i \geq 0$ and $e_i \leq a_i - x_i$;

(ii) if $x_i \geq a_i$, then agent i can only buy, in which case $e_i \leq 0$ and $e_i = a_i - x_i$.

Let λ^* continue to represent the price for a unit of shared resource. Denote $\mathbf{e} = (e_1 \dots e_n)^\top$ as the vector representing the traded resource profile across the network.

Definition 3. A triplet of price-allocation-trade profile $(\lambda^*, \mathbf{x}^*, \mathbf{e}^*)$ is a competitive equilibrium for the MAS-SALTD if the following conditions hold:

(i) Each agent i maximizes her combined payoff at $(\mathbf{x}^*, \mathbf{e}^*)$ while meeting the physical constraint, i.e., (x_i^*, e_i^*) is an optimizer of the solution to the following constrained optimization problem:

$$\begin{aligned} \max_{x_i, e_i} \quad & f_i(x_i) + \lambda^* e_i \\ \text{s.t.} \quad & x_i + e_i \leq a_i, \\ & x_i \in \mathbb{R}^{\geq 0}, e_i \in \mathbb{R}. \end{aligned} \tag{12}$$

(ii) The total demand and supply are balanced across the network:

$$\sum_{i=1}^{n} e_i^* = 0. \tag{13}$$

Definition 4. A pair of resource allocation-trade profile $(\mathbf{x}^\star, \mathbf{e}^\star)$ is a social welfare equilibrium for the MAS-SALTD if it is an optimizer to the following optimization problem:

$$\max_{\mathbf{x}, \mathbf{e}} \quad \sum_{i=1}^{n} f_i(x_i) \tag{14}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} e_i = 0, \tag{15}$$

$$x_i + e_i \leq a_i;\ i \in \mathrm{V}, \tag{16}$$

$$x_i \in \mathbb{R}^{\geq 0}, e_i \in \mathbb{R};\ i \in \mathrm{V}. \tag{17}$$

Theorem 2. *Consider the MAS-SALTD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\geq 0}$. Then the social welfare equilibrium(s) and the competitive equilibrium(s) continue to coincide under the shared load decisions for the agents. To be precise, the following statements hold.*

(i) If $(\lambda^*, \mathbf{x}^*, \mathbf{e}^*)$ is a competitive equilibrium, then $(\mathbf{x}^*, \mathbf{e}^*)$ is a social welfare equilibrium.

(ii) If $(\mathbf{x}^\star, \mathbf{e}^\star)$ is a social welfare equilibrium, then there exists $\lambda^* \in \mathbb{R}$ such that $(\lambda^*, \mathbf{x}^\star, \mathbf{e}^\star)$ is a competitive equilibrium.

Proof. (i) The proof follows the same analysis as the proof of Theorem 1(i), where the desired connection is in place with the definitions of the optimization goals, respectively, for the competitive equilibrium and the social welfare equilibrium.

(ii) The key idea of the proof continues to be based on strong duality applied to the definition of the social welfare equilibrium, as the proof of Theorem 1. However, now the social welfare equilibrium contains additional inequality constraints; that is $x_i + e_i \leq a_i$, for all $i \in \mathrm{V}$. Inevitably, such constraints will lead to auxiliary dual variables, in addition to the dual variable related to the equality constraint $\sum e_i = 0$ if we simply repeat the proof of Theorem 1. In order to highlight the role of the dual variable corresponding to the equality constraint, and establish it as the price in the competitive equilibrium, we need a refined treatment. To this end, we define a set $\mathbb{X}_i$ for all $i \in \mathrm{V}$ in terms of the inequality constraint in (16) as $\mathbb{X}_i = \{(x_i, e_i) | x_i + e_i \leq a_i; x_i \in \mathbb{R}^{\geq 0}; e_i \in \mathbb{R}\}$. Clearly, $\mathbb{X}_i$ is a polyhedral set (see Chapter 3.4.2, Duality Theory in Bertsekas (2003)). Denoting again $f_i = -g_i$, the problem (14)–(17) can be written as:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} g_i(x_i) \\ \text{s.t.} \quad & (x_i, e_i) \in \mathbb{X}_i, i \in \mathrm{V} \\ & \sum_{i=1}^{n} e_i = 0. \end{aligned} \tag{18}$$

Let τ be the Lagrange multiplier associated with constraint $\sum_{i=1}^{n} e_i = 0$. Subsequently, we can define the dual function where the primal variables are in a polyhedral set as (Bertsekas (2003), section 5.1.6): $L^*(\tau) = \sum_{i=1}^{n} L_i^*(\tau)$, where $L_i^*(\tau) = \inf_{(x_i, e_i) \in \mathbb{X}_i} \Big(g_i(x_i) + \tau e_i\Big), i \in \mathrm{V}$. Let $(\mathbf{x}^\star, \mathbf{e}^\star)$ be a social welfare equilibrium and τ^* be the dual optimal i.e. $\tau^* \in \arg\max_{\tau \in \mathbb{R}} L^*(\tau)$. Since the problem (18) is feasible and its optimal value is finite, strong duality holds (Bertsekas (2003), Proposition 5.2.1). This means that

$$\sum_{i=1}^{n} g_i(x_i^\star) = L^*(\tau^*) \tag{19}$$

$$= \sum_{i=1}^{n} \left(\inf_{(x_i, e_i) \in \mathbb{X}_i} \Big(g_i(x_i) + \tau^* e_i \Big) \right) \tag{20}$$

$$\leq \sum_{i=1}^{n} g_i(x_i^\star) + \tau^* \sum_{i=1}^{n} e_i^\star \tag{21}$$

$$\leq \sum_{i=1}^{n} g_i(x_i^\star). \tag{22}$$

Eq. (19) states that the duality gap is zero, (20) comes from the definition of the dual function, (21) follows since the minimization of $\sum_{i=1}^{n} g_i(x_i)+\tau^* \sum_{i=1}^{n} e_i$ over $(x_i, e_i) \in \mathbb{X}_i$ is always less than or equal to the value at $\sum_{i=1}^{n} g_i(x_i^\star)+\tau^* \sum_{i=1}^{n} e_i^\star$, (22) follows from $\sum_{i=1}^{n} e_i^\star = 0$. We conclude that the two inequalities hold with equality which implies $(\mathbf{x}^\star, \mathbf{e}^\star)$ minimizes $\sum_{i=1}^{n} g_i(x_i)+\tau^* \sum_{i=1}^{n} e_i$ over $(x_i, e_i) \in \mathbb{X}_i$. Therefore, there holds

$$(\mathbf{x}^\star, \mathbf{e}^\star) \in \underset{\substack{(x_i,e_i)\in\mathbb{X}_i,\\ i\in\mathrm{V}}}{\arg\min} \sum_{i=1}^{n} g_i(x_i) + \tau^* \sum_{i=1}^{n} e_i. \tag{23}$$

Since (23) is separable in all $i \in \mathrm{V}$, an equivalent formulation is

$$(x_i^\star, e_i^\star) \in \underset{(x_i,e_i)\in\mathbb{X}_i}{\arg\min}\, g_i(x_i) + \tau^* e_i, \quad i \in \mathrm{V}. \tag{24}$$

Let us define the equilibrium price λ^* as $\lambda^* = -\tau^*$. It follows from (24) that $(x_i^\star, e_i^\star)$ is the solution of the following optimization problem:

$$\begin{aligned} \max \quad & f_i(x_i) + \lambda^* e_i \\ \text{s.t.} \quad & x_i + e_i \le a_i \\ & x_i \in \mathbb{R}^{\ge 0},\ e_i \in \mathbb{R}. \end{aligned} \tag{25}$$

Hence, we conclude that the triplet $(\lambda^*, \mathbf{x}^\star, \mathbf{e}^\star)$ is a competitive equilibrium. □

Remark 1. Similar results as Theorems 1 and 2 have already existed. In classic microeconomics (Mas-Colell et al., 1995), the allocation of a competitive equilibrium is shown to be Pareto optimal. In Jokić, Lazar, and Van den Bosch (2009) and Li, Chen, and Dahleh (2015), competitive equilibrium and social welfare equilibrium are proven to be in agreement. In either case, the utility functions are continuously differentiable and the proofs are based on KKT optimality conditions. It might be possible to extend the analysis in Grüne (2016) and Grüne, Kellett, and Weller (2017) to general concave utility functions based on nonsmooth KKT optimality conditions (Kanzi, 2011). Here, our proof provides a direct duality analysis and sheds light on the proof for the dynamic case, which will be presented later.

In the presence of agent trading decisions, the price λ^* under any competitive equilibrium must be non-negative, as shown in Proposition 2.

Proposition 2. *Consider the MAS-SALTD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\ge 0}$. Let $(\lambda^*, \mathbf{x}^*, \mathbf{e}^*)$ be a competitive equilibrium under the agent trading decisions. Then there always holds that $\lambda^* \ge 0$.*

Proof. Assume $\lambda^* < 0$. Then $f_i(x_i) + \lambda^* e_i$ is a strictly decreasing function with respect to e_i. Since e_i is unbounded below and upper bounded by $e_i \le a_i - x_i$, there cannot be a finite e_i^* such that $e_i^* \in \arg\max_{e_i\in\mathbb{R}} \big(f_i(x_i) + \lambda^* e_i\big)$, contradicting the definition of the competitive equilibrium. This completes the proof. □

2.3. Numerical examples

Example 1. Consider a multi-agent system with four agents who have local resource $(a_1, a_2, a_3, a_4) = (13, 14, 4, 7)$. Each agent i is associated with a utility function f_i which is represented by $f_i(x_i) = \min(k_i x_i, \beta_i)$ with $(k_1, k_2, k_3, k_4) = (21, 20, 23, 32)$ and $(\beta_1, \beta_2, \beta_3, \beta_4) = (135, 600, 130, 150)$.

(i) Let the multi-agent system be MAS-SALD. The social welfare equilibrium can be computed by numerically solving the optimization problem (3) as $\mathbf{x}^\star = (6.429, 21.23, 5.652, 4.688)^\top$, and the corresponding optimal dual variable is also obtained as $\lambda^* = 20$. Letting $\lambda^* = 20$, we then compute a competitive equilibrium that satisfies (1)–(2) as $\mathbf{x}^* = (6.429, 21.23, 5.652, 4.688)^\top$. In particular, we obtain $x_1^* = 6.429$, $x_3^* = 5.652$, and $x_4^* = 4.688$ by solving (1), and further establish $x_2^* = 21.232$ from (2). Clearly there holds $\mathbf{x}^\star = \mathbf{x}^*$, which is consistent with Theorem 1.

(ii) Let the multi-agent system be MAS-SALTD. We compute the social welfare equilibrium $(\mathbf{x}^\star, \mathbf{e}^\star)$ by solving the optimization problem (14)–(17) as

$$\mathbf{x}^\star = (6.429, 21.23, 5.652, 4.688)^\top,$$

$$\mathbf{e}^\star = (6.571, -7.23, -1.652, 2.313)^\top.$$

The optimal dual variable τ^* corresponding to the equity constraint (15) can be obtained as $\tau^* = -20$. We take $\lambda^* = -\tau^* = 20$ and establish a competitive equilibrium that satisfies (12)–(13) as

$$\mathbf{x}^* = (6.429, 21.23, 5.652, 4.688)^\top,$$

$$\mathbf{e}^* = (6.571, -7.23, -1.652, 2.313)^\top.$$

In particular, we compute $(x_1^*, e_1^*) = (6.429, 6.571)$, $(x_3^*, e_3^*) = (5.652, -1.652)$ and $(x_4^*, e_4^*) = (4.688, 2.313)$ by solving (12), and obtain $(x_2^*, e_2^*) = (21.23, -7.23)$ from (13). Again there holds $(\mathbf{x}^\star, \mathbf{e}^\star) = (\mathbf{x}^*, \mathbf{e}^*)$, which validates Theorem 2. □

Example 2. Consider a multi-agent system with four agents. The utility function for agent i is in the quadratic form $f_i = -\frac{1}{2} b_i x_i^2 + k_i x_i$ for $i = 1, 2, 3, 4$. We consider two pairs of system parameters

$$\mathbf{b} = (2, 5, 3, 4)^\top \quad \mathbf{k} = (21, 17, 23, 13)^\top; \tag{PM.1}$$

$$\mathbf{b}' = (2, 5, 3, 4)^\top \quad \mathbf{k}' = (25, 22, 24, 14)^\top. \tag{PM.2}$$

Let the network resource capacity $C = \sum_{i=1}^{4} a_i$ take values in an interval $(0, 40)$. We get a discretization of the interval $(0, 40)$ with a step-size 0.8 that consists of 50 equidistant points for C. For each C, we compute the optimal prices of the system under MAS-SALD and MAS-SALTD.

For MAS-SALD, the optimal dual variables $\lambda_{\mathrm{SALD}}^{*(\mathrm{PM.1})}$ and $\lambda_{\mathrm{SALD}}^{*(\mathrm{PM.2})}$ are computed for 50 times corresponding to each value of C by solving (3), respectively, under the parameter setting (PM.1) and (PM.2). For MAS-SALTD, the optimal dual variables $\tau_{\mathrm{SALTD}}^{*(\mathrm{PM.1})}$ and $\tau_{\mathrm{SALTD}}^{*(\mathrm{PM.2})}$ related to the equity constraint (15) are also computed for 50 times corresponding to each value of C by solving (14)–(17), respectively, under the parameter setting (PM.1) and (PM.2), and then we take $\lambda_{\mathrm{SALTD}}^{*(\mathrm{PM.1})} = -\tau_{\mathrm{SALTD}}^{*(\mathrm{PM.1})}$ and $\lambda_{\mathrm{SALTD}}^{*(\mathrm{PM.2})} = -\tau_{\mathrm{SALTD}}^{*(\mathrm{PM.2})}$. In Fig. 1, we plot the 50 points of optimal prices versus C, to obtain an approximate trajectory of the optimal price as a function of C.

From Fig. 1 we observe that the optimal price $\lambda_{\mathrm{SALD}}^*$ in MAS-SALD can indeed take negative values; while the optimal price $\lambda_{\mathrm{SALTD}}^*$ in MAS-SALTD is always non-negative. These observations are consistent with Propositions 1 and 2. Moreover, for both MAS-SALD and MAS-SALTD, we observe in Fig. 1 that the optimal prices $\lambda_{\mathrm{SALD}}^*$, $\lambda_{\mathrm{SALTD}}^*$ are decreasing as the network resource capacity C increases. □

3. Social shaping for competitive equilibrium

Consistent with classical welfare economics theory, a competitive equilibrium, despite being a social welfare equilibrium as well, indicates nothing about fairness or sustainability. If the optimal pricing λ^* is too high, agents might opt out of the system, instead of participating in the self-sustained multi-agent system. When members leave the system, the achievable payoff for the remaining agents would go down. Therefore, the agents share a social responsibility in shaping their utility functions so that λ^* is within a socially acceptable range.

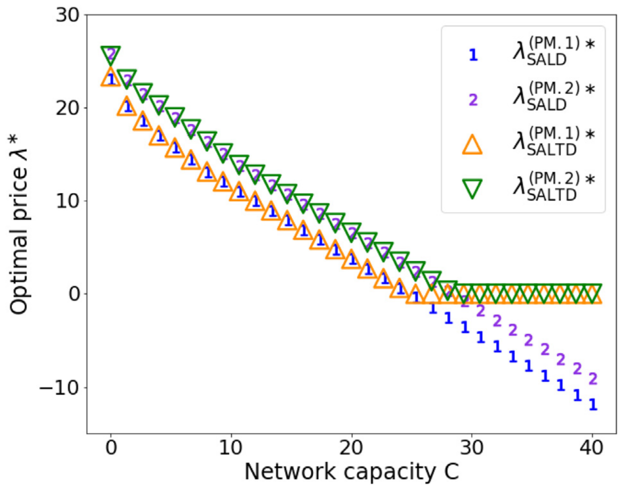

Fig. 1. The curves of the optimal prices as functions of the network resource capacity in Example 2.

3.1. Shaping the competitive equilibrium

Here we present an approach to achieve a socially acceptable competitive equilibrium, by synthesizing a class of utility functions from which agents can select. We make the following assumption.

Assumption 1. Each f_i is represented by $f_i(x_i) = -\frac{1}{2}b_i x_i^2 + k_i x_i$, where $b_i \in \mathbb{R}^{>0}$ and $k_i \in \mathbb{R}^{\geq 0}$. A utility function f_i is socially admissible if there hold $k_i \in [k_{\min}, k_{\max}]$ and $b_i \in [b_{\min}, b_{\max}]$.

Let $\lambda^\dagger$ represent the highest pricing for λ^* that agents can accept, and we term such a competitive equilibrium $\lambda^* \leq \lambda^\dagger$ a *socially resilient equilibrium*. Let $\mathbf{a} = (a_1 \dots a_n)^\top$ represent the network resource allocation profile, and let $C := \sum_{i=1}^n a_i$ represent the network resource capacity. Assuming C and $\mathbf{a}$ are given network characteristics, we consider the following problem of shaping the competitive equilibrium.

Problem. (Social Competitive Equilibrium Shaping) Consider the MAS-SALD. Find the range for $k_{\min}$, $k_{\max}$, $b_{\min}$, $b_{\max}$ under which there always exists a competitive equilibrium that leads to $\lambda^* \leq \lambda^\dagger$, for all socially admissible utility functions.

3.2. Socially admissible utility functions

Denote $\mathbf{k} = (k_1, \dots, k_n)$ and $\mathbf{b} = (b_1, \dots, b_n)^\top$. For two vectors $\mathbf{l} = (l_1, \dots, l_n)$ and $\mathbf{l}' = (l'_1, \dots, l'_n)$, we write $\mathbf{l} \preceq \mathbf{l}'$ if there holds $l_i \leq l'_i$ for all $i \in$ V. In other words, $\preceq$ defines a partial order for all vectors in $\mathbb{R}^n$.

Define

$$\mathscr{S}_* := \Big\{ \big(k_{\min}, k_{\max}, b_{\min}, b_{\max}\big) \in \mathbb{R}^4_{\geq 0} : \underbrace{\frac{nk_{\max}}{b_{\min}} - C \leq \frac{nk_{\min}}{b_{\max}}}_{con1}; \underbrace{\frac{nk_{\max}}{b_{\min}} - C \leq \frac{n\lambda^\dagger}{b_{\max}}}_{con2} \Big\}. \quad (26)$$

We present the following lemma and theorem.

Lemma 1. *Consider the MAS-SALD. Let Assumption 1 hold. The optimal load allocation x_i^*, which is an optimal solution of the optimization problem (1), satisfies*

$$x_i^* = \begin{cases} \frac{k_i - \lambda^*}{b_i} & \text{if } \lambda^* \leq k_i, \\ 0 & \text{if } \lambda^* > k_i, \end{cases} \quad (27)$$

or, put more simply, $x_i^* = \max\{0, \frac{k_i - \lambda^*}{b_i}\}$.

Proof. The objective function for each agent $i \in$ V in the problem (1) can be written as

$$\max_{x_i \in \mathbb{R}_{\geq 0}} -\frac{1}{2} b_i x_i^2 + k_i x_i + \lambda^*(a_i - x_i).$$

Without $x_i \in \mathbb{R}^{\geq 0}$, it is strictly increasing in the interval $x_i \in [-\infty, \frac{k_i - \lambda^*}{b_i}]$, and strictly decreasing in the interval $x_i \in [\frac{k_i - \lambda^*}{b_i}, \infty]$. Since x_i takes a non-negative value, the optimal solution to the problem (1) is achieved at $x_i^* = 0$ when $\lambda^* > k_i$; and it is achieved at $x_i^* = \frac{k_i - \lambda^*}{b_i}$ when $\lambda^* \leq k_i$. The proof is now complete. □

Theorem 3. *Consider the MAS-SALD. Let Assumption 1 hold. The following statements hold.*

(i) The competitive equilibrium is unique.

(ii) The optimal price λ^* is monotone increasing under the partial order over $\mathbf{k}$.

(iii) The competitive equilibrium is always socially resilient (ie $\lambda^* \leq \lambda^\dagger$) for all socially admissible utility functions as long as $(k_{\min}, k_{\max}, b_{\min}, b_{\max}) \in \mathscr{S}_*$.

Proof. (i) According the definition of competitive equilibrium, the optimal load allocation profile $\mathbf{x}^*$ must also satisfy the supply–demand balance constraint (2). Substituting the expression (27) for x_i^* into the constraint (2), we obtain

$$\sum_{i=1}^n \max\{0, \frac{k_i - \lambda^*}{b_i}\} = \sum_{i=1}^n a_i. \quad (28)$$

It is straightforward to know that the competitive equilibrium is unique.

(ii) As k_i increases, λ^* must also increase so as to compensate for the change — ensuring Eq. (28) holds. Otherwise, the left-hand side of Eq. (28) would increase, while the right-hand side remains constant, leading to Eq. (28) to not hold. Therefore, λ^* is monotone increasing under the partial order over $\mathbf{k}$.

(iii) Without loss of generality, we suppose $k_1 \geq \dots \geq k_m \geq \lambda^* \geq k_{m+1} \geq \dots \geq k_n, m \in \{1, \dots, n-1\}$, or $k_1 \geq \dots \geq k_m \geq \lambda^*, m = n$. Denote $b^m_{\min} = \min\{b_1, \dots, b_m\} \geq b_{\min}$ and $b^m_{\max} = \max\{b_1, \dots, b_m\} \leq b_{\max}$.

Let $(\lambda^*, \mathbf{x}^*)$ be a competitive equilibrium. We can write Eq. (28) as

$$\sum_{i=1}^m \frac{k_i - \lambda^*}{b_i} + \sum_{i=m+1}^n 0 = C$$

where $C = \sum_{i=1}^n a_i$ and $\lambda^* \leq k_i, \forall i \leq m$. Then we obtain

$$\lambda^* = \Big(\sum_{i=1}^m \frac{k_i}{b_i} - C\Big) / \Big(\sum_{i=1}^m \frac{1}{b_i}\Big) \quad (29)$$

$$\leq \Big(\frac{mk_{\max}}{b^m_{\min}} - C\Big) / \Big(\frac{m}{b^m_{\max}}\Big) \quad (30)$$

$$\leq \Big(\frac{nk_{\max}}{b_{\min}} - C\Big) / \Big(\frac{n}{b_{\max}}\Big) \quad (31)$$

$$\leq \lambda^\dagger \quad (32)$$

where $k_{\max} = k_1$. The second inequality from (30) to (31) is guaranteed by $m \leq n, b^m_{\min} \geq b_{\min}$ and $b^m_{\max} \leq b_{\max}$. The third inequality from (31) to (32) is guaranteed by *con2* in the set $\mathscr{S}_*$.

We also know

$$\lambda^* \leq \Big(\frac{nk_{\max}}{b_{\min}} - C\Big) / \Big(\frac{n}{b_{\max}}\Big)$$

$$\leq k_{\min} \quad (33)$$

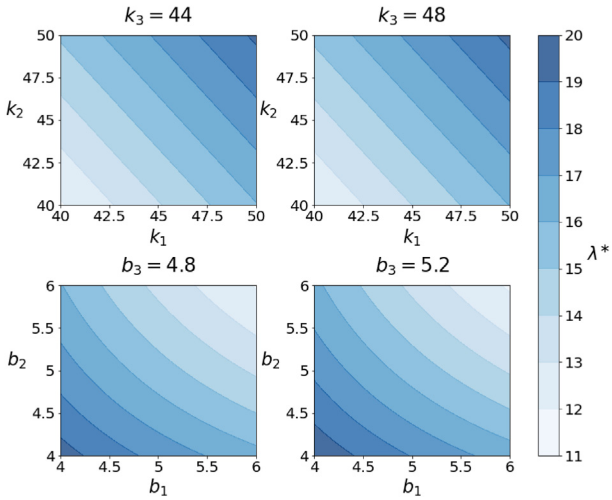

Fig. 2. Contour maps for the optimal prices in Example 3.

$$\leq k_{\min}^m \tag{34}$$

$$\leq k_i, \forall i \leq m \tag{35}$$

The inequality from (32) to (33) is upon *con1* in the set $\mathscr{S}_*$. The remaining inequalities is guaranteed by $k_1 \geq \cdots \geq k_m \geq \cdots \geq k_n$, $m \in \{1, \ldots, n-1\}$, or $k_1 \geq \cdots \geq k_m \geq, m = n$.

Collecting all conditions for $(k_{\min}, k_{\max}, b_{\min}, b_{\max})$, we obtain that $\lambda^* \leq \lambda^\dagger$ for all socially admissible utility functions with $(k_{\min}, k_{\max}, b_{\min}, b_{\max}) \in \mathscr{S}_*$. $\square$

Remark 2. In this manuscript, the pricing threshold $\lambda^\dagger$ that can be socially accepted by all agents is given as a system requirement. Agents can find the competitive equilibrium at which the optimal pricing is below a threshold in a distributed way. Each agent chooses its parameters based on (26), communicates its parameters and local resource with other agents, solves a social welfare equilibrium, and identifies the corresponding Lagrange multiplier as the optimal pricing λ^* which is guaranteed to be socially acceptable by Theorem 3. For more details, we refer to Salehi, Chen, Petersen, Ratnam, and Shi (2021).

3.3. Numerical examples

Example 3. Consider a MAS-SALD with three agents and network capacity $C = 18$. Each agent's utility function is set as the quadratic form $f_i = -\frac{1}{2} b_i x_i^2 + k_i x_i$ for $i = 1, 2, 3$. The system's highest pricing for λ^* that agents can accept socially is assumed to be $\lambda^\dagger = 39$. Take $b_{\min} = 4$, $b_{\max} = 6$, $k_{\min} = 40$, and $k_{\max} = 50$. We can verify such a configuration of $(b_{\min}, b_{\max}, k_{\min}, k_{\max})$ is a point in $\mathscr{S}_*$ defined in (26).

(i) Let $\mathbf{b}$ be fixed to be $\mathbf{b} = (4, 5, 6)^\top$. Take $k_3 \in \{44, 48\}$. We discretize the space for $(k_1, k_2) \in [40, 50]^2$ and compute the optimal pricing λ^* by solving the optimal dual variable of (3). Then we plot the contour maps for the optimal price as a function of k_1 and k_2 in the first row of Fig. 2.

(ii) Let $\mathbf{k}$ be fixed to be $\mathbf{k} = (44, 46, 48)^\top$. Take $b_3 \in \{4.8, 5.2\}$. We discretize the space for $(b_1, b_2) \in [4, 6]^2$ and compute the optimal pricing λ^* by solving the optimal dual variable of (3). Then we plot the contour maps for the optimal price as a function of b_1 and b_2 in the second row of Fig. 2.

In Fig. 2 we observe that the maximum value for the price λ^* is 20, which is lower than $\lambda^\dagger = 39$. This illustrates all socially admissible utility functions for parameters in the set $\mathscr{S}_*$ lead to socially acceptable prices, providing a validation for Theorem 3(ii). From the first row of Fig. 2, the optimal price is monotone under the partial order $\preceq$ with respect to $\mathbf{k}$, which is consistent with Theorem 3(iii). $\square$

4. Dynamic multi-agent systems

4.1. MAS with Dynamic Agent Load/Trading Decisions

Here we consider the load balancing problem for dynamical multi-agent systems.

MAS with Dynamic Agent Load/Trading Decisions (MAS-DALTD). Each agent $i \in \mathrm{V}$ is associated with a dynamical state $\mathbf{y}_i(t) \in \mathbb{R}^m$, described by

$$\mathbf{y}_i(t+1) = \mathbf{A}_i \mathbf{y}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \quad t \in \mathcal{T}, \tag{36}$$

where $\mathbf{u}_i(t) \in \mathbb{R}^d$ is the control input, and $\mathbf{A}_i$ and $\mathbf{B}_i$ are real matrices with proper dimensions. The time steps are indexed by $\mathcal{T} = \{0, \ldots, T-1\}$. Associated with $t \in \mathcal{T}$, agent i incurs a utility function $f_i(\mathbf{y}_i(t), \mathbf{u}_i(t))$; the terminal utility for agent i is $\Phi_i(\mathbf{y}_i(T))$. Upon taking the control action $\mathbf{u}_i(t)$, the required resource is defined by the function $h_i(\mathbf{u}_i(t))$. In the context of thermostatically controlled loads, we may interpret $\mathbf{y}_i$ and $\mathbf{u}_i$ as energy states and temperature setponits of all load units for agent i, respectively. Each agent can produce an $a_i(t)$ units of resource at time t, and also makes a trading decision $e_i(t)$ units of resource over the network at time t. Similarly,

(i) if $h_i(\mathbf{u}_i(t)) < a_i(t)$, then agent i can sell, in which case $e_i(t) \geq 0$ and $e_i(t) \leq a_i(t) - h_i(\mathbf{u}_i(t))$;
(ii) if $h_i(\mathbf{u}_i(t)) \geq a_i(t)$, then agent i will buy, in which case $e_i(t) \leq 0$ and $e_i(t) = a_i(t) - h_i(\mathbf{u}_i(t))$.

We denote $\boldsymbol{\lambda} = (\lambda_0 \ldots \lambda_{T-1})^\top$ as the pricing vector through the time horizon, where λ_t is the unit price for traded energy at step t. Consequently, the payoff of agent i throughout $[0, T]$ is described by

$$\sum_{t=0}^{T-1} \Big(f_i(\mathbf{y}_i(t), \mathbf{u}_i(t)) + \lambda_t e_i(t) \Big) + \Phi_i(\mathbf{y}_i(T)).$$

We denote $\mathbf{y}(t) = (\mathbf{y}_1(t)^\top \ldots \mathbf{y}_n(t)^\top)^\top$, $\mathbf{u}(t) = (\mathbf{u}_1(t)^\top \ldots \mathbf{u}_n(t)^\top)^\top$, and $\mathbf{e}(t) = (\mathbf{e}_1(t)^\top \ldots \mathbf{e}_n(t)^\top)^\top$. We further define $\mathbf{Y} = (\mathbf{y}(0)^\top \ldots \mathbf{y}(T)^\top)^\top$, $\mathbf{U} = (\mathbf{u}(0)^\top \ldots \mathbf{u}(T-1)^\top)^\top$ and $\mathbf{E} = (\mathbf{e}(0)^\top \ldots \mathbf{e}(T-1)^\top)^\top$. Also introduce $\mathbf{U}_i = (\mathbf{u}_i(0)^\top \ldots \mathbf{u}_i(T-1)^\top)^\top$, $\mathbf{E}_i = (\mathbf{e}_i(0)^\top \ldots \mathbf{e}_i(T-1)^\top)^\top$ and $\mathbf{a}_i = (a_i(0)^\top \ldots a_i(T-1)^\top)^\top$.

Definition 5. Let $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^{mn}$ be given. A triple of price-control-trading profiles $(\boldsymbol{\lambda}^*, \mathbf{U}^*, \mathbf{E}^*)$ is a dynamic competitive equilibrium if the following conditions hold:

(i) each agent i maximizes its combined payoff under $\mathbf{U}_i^*$ and $\mathbf{E}_i^*$:

$$\begin{aligned} \max_{\mathbf{U}_i, \mathbf{E}_i} \quad & \sum_{t=0}^{T-1} \Big(f_i(\mathbf{y}_i(t), \mathbf{u}_i(t)) + \lambda_t^* e_i(t) \Big) + \Phi_i(\mathbf{y}_i(T)) \\ \text{s.t.} \quad & \mathbf{y}_i(t+1) = \mathbf{A}_i \mathbf{y}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \quad t \in \mathcal{T}, \\ & e_i(t) \leq a_i(t) - h_i(\mathbf{u}_i(t)), \quad t \in \mathcal{T}; \end{aligned} \tag{37}$$

(ii) the total demand and supply are balanced across the network for all time, i.e., there holds

$$\sum_{i=1}^{n} e_i(t) = 0, \quad t \in \mathcal{T}. \tag{38}$$

Definition 6. Let $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^{mn}$ be given. A pair of control-trading profiles $(\mathbf{U}^\star, \mathbf{E}^\star)$ is a dynamic social welfare equilibrium if it is a solution to the following optimal control problem:

$$\max_{\mathbf{U},\mathbf{E}} \quad \sum_{i=1}^{n}\Big(\sum_{t=0}^{T-1} f_i(\mathbf{y}_i(t),\mathbf{u}_i(t)) + \Phi_i(\mathbf{y}_i(T))\Big) \tag{39}$$

$$\text{s.t.} \quad \mathbf{y}_i(t+1) = \mathbf{A}_i\mathbf{y}_i(t) + \mathbf{B}_i\mathbf{u}_i(t), \quad t\in\mathcal{T},\ i\in\mathrm{V}, \tag{40}$$

$$e_i(t) \le a_i(t) - h_i(\mathbf{u}_i(t)), \quad t\in\mathcal{T},\ i\in\mathrm{V}, \tag{41}$$

$$\sum_{i=1}^{n} e_i(t) = 0, \quad t\in\mathcal{T}. \tag{42}$$

4.2. Dynamic competitive equilibrium

We impose the following assumption.

Assumption 2. (i) the Φ_i are concave functions for $i\in\mathrm{V}$; (ii) the f_i are concave functions for $i\in\mathrm{V}$; (iii) the h_i are nonnegative convex functions for $i\in\mathrm{V}$, and $h_i(\mathbf{z})<b$ defines a bounded open set of $\mathbf{z}$ in $\mathbb{R}^d$ for $b>0$; (iv) $\sum_{i=1}^n a_i(t)>0$ for all $t\in\mathcal{T}$.

We present the following result which establishes a similar connection between the competitive equilibrium and social welfare equilibrium under this dynamic setting.

Theorem 4. *Consider the MAS-DALTD with $\mathbf{y}(0)=\mathbf{y}_0\in\mathbb{R}^{mn}$ be given. Let Assumption 2 hold. The dynamic social welfare equilibrium(s) and the dynamic competitive equilibrium(s) coincide and the following statements hold.*

(i) If $(\boldsymbol{\lambda}^*,\mathbf{U}^*,\mathbf{E}^*)$ is a dynamic competitive equilibrium, then $(\mathbf{U}^*,\mathbf{E}^*)$ is a dynamic social welfare equilibrium.

(ii) If $(\mathbf{U}^\star,\mathbf{E}^\star)$ is a dynamic social welfare equilibrium, then there exists $\boldsymbol{\lambda}^*\in\mathbb{R}^T$ such that $(\boldsymbol{\lambda}^*,\mathbf{U}^\star,\mathbf{E}^\star)$ is a competitive equilibrium.

Proof. (i) The proof of sufficiency follows from a similar analysis as in the proof of Theorem 1. The transition from competitive equilibrium to social welfare equilibrium under this dynamical setting continues to be a direct consequence of the formulations of the two underlying optimization problems.

(ii) First of all, the dynamics $\mathbf{y}_i(t+1)=\mathbf{A}_i\mathbf{y}_i(t)+\mathbf{B}_i\mathbf{u}_i(t)$ with given $\mathbf{y}(0)$ ensures that any $\mathbf{y}_i(t)$ for $t=1,\dots,T$ is a linear combination of $\mathbf{y}(0)$ and $\mathbf{u}_i(0),\dots,\mathbf{u}_i(t-1)$. Therefore, we can always write for any $i\in\mathrm{V}$ that

$$\mathbf{y}_i(t) = p_{i,t}\mathbf{y}_0 + q_{i,t}\mathbf{U}_i,\ \ t=0,\dots,T \tag{43}$$

with $p_{i,t}$ and $q_{i,t}$ being matrices with proper dimensions. As a result, we have

$$f_i(\mathbf{y}_i(t),\mathbf{u}_i(t)) = f_i(p_t\mathbf{y}_0 + q_t\mathbf{U}_i,\mathbf{u}_i(t)) := \tilde{f}_{i,t}(\mathbf{U}_i). \tag{44}$$

In view of Assumption 2, and the fact that the composition of a concave function and an affine function continues to be concave, we conclude that $g_{i,t}(\cdot)$ is a concave function. Similarly,

$$\Phi(\mathbf{y}_i(t)) = \Phi(p_{i,T}\mathbf{y}_0 + q_{i,T}\mathbf{U}_i) := \Phi_i(\mathbf{U}_i)$$

where $\Phi_i(\cdot)$ is a concave function.

The optimization problem (39)–(42) can be equivalently rewritten as the following convex programming problem:

$$\begin{aligned}\min_{\mathbf{U},\mathbf{E}} \quad & -\sum_{i=1}^{n}\Big(\sum_{t=0}^{T-1}\tilde{f}_{i,t}(\mathbf{U}_i) + \Phi_i(\mathbf{U}_i)\Big)\\ \text{s.t.} \quad & h_i(\mathbf{u}_i(t)) + e_i(t) \le a_i(t), \quad t=0,\dots,T-1,\ i\in\mathrm{V}\\ & \sum_{i=1}^{n} e_i(t) = 0, \quad t=0,\dots,T-1.\end{aligned} \tag{45}$$

Similarly, (37) can be equivalently written as the following convex programme

$$\begin{aligned}\min_{\mathbf{U}_i,\mathbf{E}_i} \quad & -\sum_{t=0}^{T-1}\Big(\tilde{f}_{i,t}(\mathbf{U}_i) + \lambda_t^* e_i(t)\Big) + \Phi_i(\mathbf{U}_i)\\ \text{s.t.} \quad & e_i(t) \le a_i(t) - h_i(\mathbf{u}_i(t)),\ t=0,\dots,T-1.\end{aligned} \tag{46}$$

Next, with Assumption 2.(iii)–(iv), we can verify that Slater's condition holds for (45) and (46), which guarantees strong duality in both problems (Boyd et al., 2004). Also, noting

$$\sum_{i=1}^{n} h_i(\mathbf{u}_i(t)) \le \sum_{i=1}^{n} a_i(t) \tag{47}$$

and the assumption that $h_i(\mathbf{z})<b$ defines a bounded open set of $\mathbf{z}$ in $\mathbb{R}^m$ for $b>0$, $\mathbf{u}_i(t)$ takes values in a compact set for all i and t. Moreover, since $h_i(\mathbf{u}_i(t))\ge 0$ holds for all i and for all t, there holds $e_i(t)\le a_i(t)$. The constraint $\sum_{i=1}^n e_i(t)=0$ further ensures $e_i(t)\ge -\sum_{i=1}^n a_i(t)$ for all i and t. Thus $e_i(t)$ also takes values in a compact set for all i and t. The convex programming problem (45) leads to finite primal solution.

The Lagrange dual function of (45) can be written as

$$\begin{aligned}L(\mathbf{U},\mathbf{E},\boldsymbol{\lambda},\boldsymbol{\mu}) &= -\sum_{i=1}^{n}\Big(\sum_{t=0}^{T-1}\tilde{f}_{i,t}(\mathbf{U}_i) + \Phi_i(\mathbf{U}_i)\Big)\\ &\quad + \sum_{t=0}^{T-1}\sum_{i=1}^{n}\lambda_t e_i(t)\\ &\quad + \sum_{t=0}^{T-1}\sum_{i=1}^{n}\mu_{i,t}\big(h_i(\mathbf{u}_i(t)) + e_i(t) - a_i(t)\big)\\ &= \sum_{i=1}^{n} L_i(\mathbf{U}_i,\mathbf{E}_i,\boldsymbol{\lambda},\boldsymbol{\mu}_i)\end{aligned} \tag{48}$$

where

$$\begin{aligned}L_i(\mathbf{U}_i,\mathbf{E}_i,\boldsymbol{\lambda},\boldsymbol{\mu}_i) &= -\sum_{t=0}^{T-1}\tilde{f}_{i,t}(\mathbf{U}_i) + \Phi_i(\mathbf{U}_i) + \sum_{t=0}^{T-1}\lambda_t e_i(t)\\ &\quad + \sum_{t=0}^{T-1}\mu_{i,t}\big(h_i(\mathbf{u}_i(t)) + e_i(t) - a_i(t)\big).\end{aligned} \tag{49}$$

Here $\mu_{i,t}\ge 0$ since they correspond to the inequality constraints. We have used the conventional notation $\boldsymbol{\mu}_i=(\mu_{i,0},\dots,\mu_{i,T-1})^\top$ and $\boldsymbol{\mu}=(\boldsymbol{\mu}_1^\top,\dots,\boldsymbol{\mu}_n^\top)^\top$.

Finally, letting an optimal dual solution of (45) be $(\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)$, there holds from strong duality

$$(\mathbf{U}^\star,\mathbf{E}^\star) \in \arg\min\ L(\mathbf{U},\mathbf{E},\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*) \tag{50}$$

if $(\mathbf{U}^\star,\mathbf{E}^\star)$ is a dynamic social welfare equilibrium. This implies from (48) that

$$(\mathbf{U}_i^\star,\mathbf{E}_i^\star) \in \arg\min\ L_i(\mathbf{U},\mathbf{E},\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*). \tag{51}$$

Now, $\boldsymbol{\mu}^*$ is obtained by solving

$$\max_{\boldsymbol{\lambda},\boldsymbol{\mu}}\ \min_{\mathbf{U},\mathbf{E}}\ L(\mathbf{U},\mathbf{E},\boldsymbol{\lambda},\boldsymbol{\mu}) = \max_{\boldsymbol{\lambda},\boldsymbol{\mu}}\ \min_{\mathbf{U},\mathbf{E}}\ \sum_{i=1}^{n} L_i(\mathbf{U}_i,\mathbf{E}_i,\boldsymbol{\lambda},\boldsymbol{\mu}_i) \tag{52}$$

where the maximization and minimization are taken in their respective domains for $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{U}$, $\mathbf{E}$. As a result, there must hold

$$\boldsymbol{\mu}_i^* \in \arg\max_{\boldsymbol{\mu}_i}\min_{\mathbf{U}_i,\mathbf{E}_i} L_i(\mathbf{U}_i,\mathbf{E}_i,\boldsymbol{\lambda}^*,\boldsymbol{\mu}_i). \tag{53}$$

It is worth emphasizing that $L_i(\mathbf{U}_i,\mathbf{E}_i,\boldsymbol{\lambda}^*,\boldsymbol{\mu}_i)$ is, precisely, the Lagrangian of (46). Therefore, (53) ensures that $\boldsymbol{\mu}_i^*$ is an optimal dual solution of (46), and then from strong duality (51) ensures

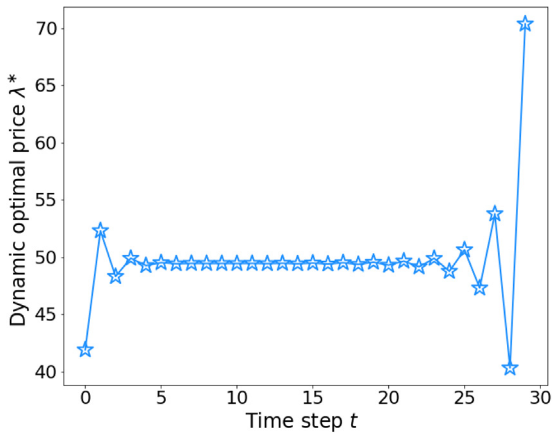

Fig. 3. The dynamic optimal price vs. time steps in Example 4.

that $(\mathbf{U}^\star, \mathbf{E}^\star)$ is an optimal primal solution of (46). In other words, we have proven $(\boldsymbol{\lambda}^*, \mathbf{U}^\star, \mathbf{E}^\star)$ is a competitive equilibrium.

The proof of the theorem is now complete. □

Example 4. Let the time horizon be $T = 30$. Consider a MAS-DALTD with three agents which have initial states

$$\mathbf{y}_1(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \mathbf{y}_2(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{y}_3(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix},$$

and local resource

$$\mathbf{a}_1 = \mathbf{a}_2 = [50 \cdots 50]_{30\times 1}, \quad \mathbf{a}_3 = [30 \cdots 30]_{30\times 1}.$$

The dynamical state $\mathbf{y}_i(t) \in \mathbb{R}^2$ of agent i is described by

$$\mathbf{y}_i(t+1) = \mathbf{A}_i\mathbf{y}_i(t) + \mathbf{B}_i\mathbf{u}_i(t), \quad t \in \mathcal{T},$$

where

$$\mathbf{A}_1 = \begin{bmatrix} \frac{-3}{5} & 0 \\ 0 & \frac{-7}{10} \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} \frac{-1}{2} & 0 \\ 0 & \frac{-1}{5} \end{bmatrix}, \mathbf{A}_3 = \begin{bmatrix} \frac{-2}{5} & 0 \\ 0 & \frac{-4}{5} \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}, \mathbf{B}_3 = \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix}.$$

The utility function of agent i is in the quadratic form

$$f_i = \mathbf{y}_i^\top(t)\mathbf{R}_i\mathbf{y}_i(t) + \mathbf{W}_i\mathbf{y}_i(t) + \mathbf{u}_i^\top(t)\mathbf{Q}_i\mathbf{u}_i(t) + \mathbf{K}_i\mathbf{u}_i(t),$$

where

$$\mathbf{R}_1 = \begin{bmatrix} -5 & 0 \\ 0 & -8 \end{bmatrix}, \mathbf{R}_2 = \begin{bmatrix} -3 & 0 \\ 0 & -7 \end{bmatrix}, \mathbf{R}_3 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\mathbf{Q}_1 = \begin{bmatrix} -5 & 0 \\ 0 & -4 \end{bmatrix}, \mathbf{Q}_2 = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}, \mathbf{Q}_3 = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix},$$

$$\mathbf{W}_1 = \begin{bmatrix} 200 & 300 \end{bmatrix}, \mathbf{W}_2 = \begin{bmatrix} 200 & 400 \end{bmatrix}, \mathbf{W}_3 = \begin{bmatrix} 450 & 300 \end{bmatrix},$$

$$\mathbf{K}_1 = \begin{bmatrix} 50 & 60 \end{bmatrix}, \mathbf{K}_2 = \begin{bmatrix} 50 & 20 \end{bmatrix}, \mathbf{K}_3 = \begin{bmatrix} 80 & 20 \end{bmatrix}.$$

The terminal utility of agent i is also set as the quadratic form $\Phi_i(\mathbf{y}_i(T)) = \mathbf{y}_i^\top(T)\mathbf{R}_i\mathbf{y}_i(T) + \mathbf{W}_i\mathbf{y}_i(T)$. Upon taking $\mathbf{u}_i(t)$, the required resource is determined by $h_i(\mathbf{u}_i(t)) = \mathbf{u}_i^\top(t)\mathbf{H}_i\mathbf{u}_i(t)$, where

$$\mathbf{H}_1 = \begin{bmatrix} 5 & 0 \\ 0 & 8 \end{bmatrix}, \mathbf{H}_2 = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}, \mathbf{H}_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

We compute the dynamic social welfare equilibrium $(\mathbf{U}^\star, \mathbf{E}^\star)$ by solving the optimization problem (39)–(42) and the optimal dual variables $-\boldsymbol{\lambda}^*$ corresponding to (42). Given $\boldsymbol{\lambda}^*$, we further compute the dynamic competitive equilibrium $(\mathbf{U}^*, \mathbf{E}^*)$ by solving (37). In Fig. 4, we plot the dynamic social welfare equilibrium and the dynamic competitive equilibrium. In Fig. 4 we observe that the dynamic social welfare equilibrium and the dynamic competitive equilibrium agree, which is consistent with Theorem 4. The dynamic optimal price for traded resource versus time steps is also shown in Fig. 3, where the price experiences oscillations both at the beginning and in the end of the time horizon, and holds a steady value in between. Turnpike properties are observed in this example that the optimal solution settled in a long time horizon approximately consists three pieces: the first and the last being transient short-time trajectory and the middle being a long-time trajectories staying close to the optimal steady-state solution of an associated static optimal control problem (Grüne et al., 2017). □

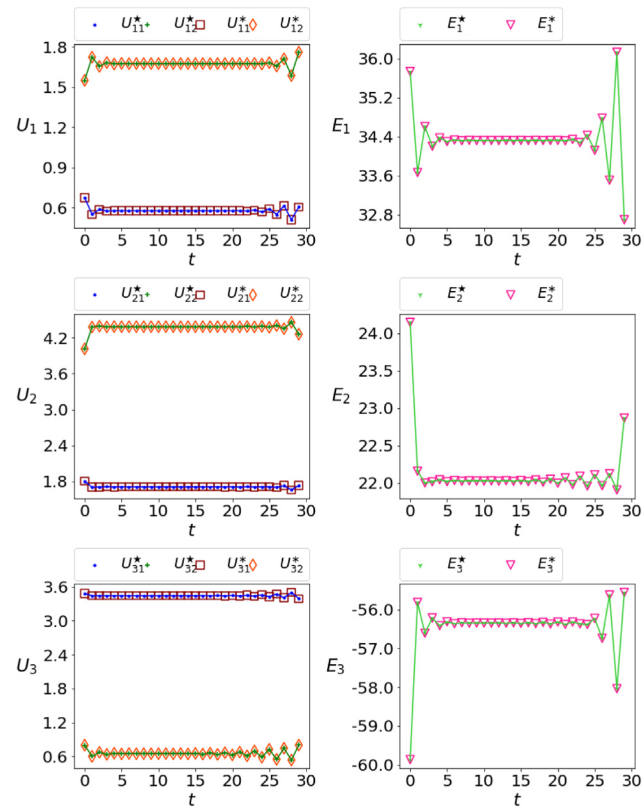

Fig. 4. The dynamic social welfare equilibrium and competitive equilibrium in Example 4.

4.3. Recursive computation of social welfare equilibrium

One of the most widely used methods in optimal control problems is dynamic programming (Bertsekas, 2003). Here we investigate the possibility of using a dynamic programming approach to represent and compute the social welfare equilibriums described in (39)–(42) as optimal feedback decisions.

Denote

$$\mathscr{E}_t := \{\mathbf{e}(t) : \sum_{i=1}^{n} e_i(t) = 0; \, e_i(t) \le a_i(t), \forall i \in V\} \tag{54}$$

and

$$\mathscr{U}_t := \{\mathbf{u}(t) : e_i(t) \le a_i(t) - h_i(\mathbf{u}_i(t)), \forall i \in V\}. \tag{55}$$

Also define $\mathbf{f}_t(\mathbf{y}(t), \mathbf{u}(t)) = -\sum_{i=1}^{n} f_i(\mathbf{y}_i(t), \mathbf{u}_i(t))$ and $\boldsymbol{\Phi}(\mathbf{y}(T)) = -\sum_{i=1}^{n} \Phi_i(\mathbf{y}_i(T))$.

Definition 7. Consider the MAS-DALTD with $\mathbf{y}(0) = \mathbf{y}_0$. Let $\mathbf{y}(t) \in \mathscr{Y}_t, \mathbf{u}(t) \in \mathscr{U}_t$, and $\mathbf{e}(t) \in \mathscr{E}_t$. We say that $(\mathbf{u}(t), \mathbf{e}(t))$ follows the feedback policy $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \dots, \boldsymbol{\pi}_{T-1})$ if there holds $(\mathbf{u}(t), \mathbf{e}(t)) = \boldsymbol{\pi}_t(\mathbf{y}(t))$, for all $t \in \mathcal{T}$.

Further introduce the cost-to-go function associated with any feedback policy $\boldsymbol{\pi}$ by

$$V^{\pi}(k, \mathbf{y}_k) := \sum_{t=k}^{T-1} \mathbf{f}_t(\mathbf{y}(t), \mathbf{u}(t)) + \boldsymbol{\Phi}(\mathbf{y}_T) \tag{56}$$

with $(\mathbf{u}(t), \mathbf{e}(t)) = \boldsymbol{\pi}_t(\mathbf{y}(t))$, for all $t = k, \dots, T-1$, $\mathbf{y}(k) = \mathbf{y}_k$, and

$$\mathbf{y}(s+1) = \mathbf{A}\mathbf{y}(s) + \mathbf{B}\mathbf{u}(s), s = k, \dots, T-1. \tag{57}$$

Here $\mathbf{A}$ and $\mathbf{B}$ are block diagonal matrices

$$\text{diag}(\mathbf{A}_0, \dots, \mathbf{A}_n) \text{ and } \text{diag}(\mathbf{B}_0, \dots, \mathbf{B}_n)$$

respectively.

It is clear that the cost-to-go function must satisfy the boundary condition that

$$V^{\pi}(T, \mathbf{y}(T)) = \boldsymbol{\Phi}(\mathbf{y}_T). \tag{58}$$

Theorem 5. *Consider the MAS-DALTD with $\mathbf{y}(0) = \mathbf{y}_0$. Let $(\mathbf{U}^\star, \mathbf{E}^\star)$ be a dynamic social welfare equilibrium from (39)–(42). Then there exists $\boldsymbol{\pi}^\star$ such that $(\mathbf{u}^\star(t), \mathbf{e}^\star(t)) = \boldsymbol{\pi}^\star_t(\mathbf{y}(t))$, for all $t \in \mathcal{T}$, and the cost-to-go function $V^{\pi^\star}$ satisfies the following recurrance equation*

$$V^{\pi^\star}(k, \mathbf{y}_k) = \min_{\substack{\mathbf{u}(k)\in\mathscr{U}_k;\\ \mathbf{e}(k)\in\mathscr{E}_k}} \Big[\mathbf{f}_k(\mathbf{y}(k), \mathbf{u}(k)) + V^{\pi^\star}(k+1, \mathbf{y}_{k+1})\Big]. \tag{59}$$

Proof. First of all, the optimization problem (39)–(42) can be rewritten as:

$$\min_{\mathbf{U},\mathbf{E}} \quad \sum_{t=0}^{T-1} \mathbf{f}_t(\mathbf{y}(t), \mathbf{u}(t)) + \boldsymbol{\Phi}(\mathbf{y}_T) \tag{60}$$

$$\text{s.t.} \quad \mathbf{y}(t+1) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}\mathbf{u}(t), \quad t \in \mathcal{T} \tag{61}$$

$$\mathbf{e}(t) \in \mathscr{E}_t, \quad t \in \mathcal{T} \tag{62}$$

$$\mathbf{u}(t) \in \mathscr{U}_t, \quad t \in \mathcal{T}. \tag{63}$$

Given the form of cost-to-go function (56) and the boundary condition of the terminal cost (58), we next have

$$V^{\pi}(T-1, \mathbf{y}_{T-1}) = \mathbf{f}_{T-1}(\mathbf{y}_{T-1}, \mathbf{u}(T-1)) + V^{\pi}(T, \mathbf{y}_T) \tag{64}$$

where $V^{\pi}(T-1, \mathbf{y}_{T-1})$ is a one-step process with initial state $\mathbf{y}_{T-1}$. The value of $V^{\pi}(T-1, \mathbf{y}_{T-1})$ depends only on $\mathbf{y}_{T-1}$ and $\mathbf{u}(T-1)$, since $\mathbf{y}_T$ is related to $\mathbf{y}_{T-1}$ and $\mathbf{u}(T-1)$ through the system dynamics (61). The optimal cost is then

$$V^{\pi^\star}(T-1, \mathbf{y}_{T-1}) \triangleq \min_{\substack{\mathbf{u}(T-1)\in\mathscr{U}_{T-1};\\ \mathbf{e}(T-1)\in\mathscr{E}_{T-1}}} \Big[\mathbf{f}_{T-1}(\mathbf{y}_{T-1}, \mathbf{u}(T-1)) + V^{\pi}(T, \mathbf{y}_T)\Big]. \tag{65}$$

where the optimal choice of $(\mathbf{u}(T-1), \mathbf{e}(T-1))$ only depends on $\mathbf{y}_{T-1}$. The cost over the last two intervals is given by

$$V^{\pi}(T-2, \mathbf{y}_{T-2}) = \mathbf{f}_{T-2}(\mathbf{y}_{T-2}, \mathbf{u}(T-2)) + V^{\pi}(T-1, \mathbf{y}_{T-1}) \tag{66}$$

where $V^{\pi}(T-2, \mathbf{y}_{T-2})$ is a two-step process with initial state $\mathbf{y}_{T-2}$. The optimal policy during these two steps is found from

$$V^{\pi^\star}(T-2, \mathbf{y}_{T-2}) \triangleq \min_{\substack{\mathbf{u}(T-2)\in\mathscr{U}_{T-2};\\ \mathbf{e}(T-2)\in\mathscr{E}_{T-2}}} \Big[V^{\pi}(T-1, \mathbf{y}_{T-1}) + \mathbf{f}_{T-2}(\mathbf{y}_{T-2}, \mathbf{u}(T-2))\Big]. \tag{67}$$

The optimality principle states that for this two-step process, whatever the initial state $\mathbf{y}_{T-2}$, initial control action and trading decision $(\mathbf{u}(T-2), \mathbf{e}(T-2))$, the remaining $(\mathbf{u}(T-1), \mathbf{e}(T-1))$ must be optimal with respect to $\mathbf{y}_{T-1}$ resulted by applying $(\mathbf{u}(T-2), \mathbf{e}(T-2))$ to the system; that is,

$$V^{\pi^\star}(T-2, \mathbf{y}_{T-2}) \triangleq \min_{\substack{\mathbf{u}(T-2)\in\mathscr{U}_{T-2};\\ \mathbf{e}(T-2)\in\mathscr{E}_{T-2}}} \Big[V^{\pi^\star}(T-1, \mathbf{y}_{T-1}) + \mathbf{f}_{T-2}(\mathbf{y}_{T-2}, \mathbf{u}(T-2))\Big]. \tag{68}$$

Applying the same analysis recursively, we obtain

$$V^{\pi^\star}(k, \mathbf{y}_k) = \min_{\substack{\mathbf{u}(k)\in\mathscr{U}_k;\\ \mathbf{e}(k)\in\mathscr{E}_k}} \Big[\mathbf{f}_k(\mathbf{y}_k, \mathbf{u}(k)) + V^{\pi^\star}(k+1, \mathbf{y}_{k+1})\Big]. \tag{69}$$

Now let $(\mathbf{U}^\star, \mathbf{E}^\star)$ be a dynamic social welfare equilibrium. The value of $y(t+1)$ only depends on the current state $y(t)$ and the current input $u(t)$, but not on the historical states and inputs before t. Therefore, the open-loop optimal solution $(\mathbf{U}^\star, \mathbf{E}^\star)$ of (39)–(42) coincides with the feedback optimal policies $\boldsymbol{\pi}^\star$. The proof of the theorem is now complete. □

4.4. Social smoothing via receding horizon pricing

From Example 4, it is evident that dynamic multi-agent systems operating under competitive equilibriums for a fixed horizon may encounter significant pricing oscillations, especially at the beginning and towards the end of the time period. In practice, this means users are experiencing market shocks, which is not desirable from a social point of view. Therefore, for dynamic multi-agent systems, socially resilient competitive equilibriums should have pricing trajectories that are as smooth as possible. In addition, computing the dynamic competitive equilibriums over a long period of time is also a challenging task, and may even be infeasible for large-scale multi-agent systems.

The receding-horizon approach (Mayne, 2014; Qin & Badgwell, 2003) is a proven method for delivering robust and computationally efficient controllers for dynamical systems, with successful applications in a wide range of areas ranging from emergency vehicle scheduling (Goodwin & Medioli, 2013) to dynamic hedging of options (Bemporad, Bellucci, & Gabbriellini, 2014). The control input trajectories derived from a receding-horizon approach may even be good approximations of the optimal control solution under suitable conditions (Grüne, 2016). With this view, we next propose a receding horizon pricing procedure for the considered dynamic multi-agent systems.

Consider the MAS-DALTD with $\mathbf{y}(0) = \mathbf{y}_0$. We fix a prediction horizon N and denote $\mathcal{K} = \{0, 1, \dots, N-1\}$. The receding horizon approach approximates the solution to the optimization problem of (39)–(42) as follows. Assume a full measurement of the estimate of the state $\mathbf{y}_i(t)$, $i \in \mathrm{V}$, is available at each time step $t, t \in \mathcal{T}$. We then propose a new optimization problem over the horizon $[t, t+N]$ at each time step $t, t \in \mathcal{T}$:

$$\min_{\substack{\mathbf{U}_{t\to t+N|t}\\ \mathbf{E}_{t\to t+N|t}}} \quad \boldsymbol{\Phi}(\mathbf{y}_{t+N|t}) + \sum_{k=0}^{N-1} \mathbf{f}_k(\mathbf{y}_{t+k|t}, \mathbf{u}_{t+k|t}) \tag{70}$$

$$\text{s.t.} \quad \mathbf{y}_{i,t+k+1|t} = \mathbf{A}_i\mathbf{y}_{i,t+k|t} + \mathbf{B}_i\mathbf{u}_{i,t+k|t}, k \in \mathcal{K}; i \in \mathrm{V}, \tag{71}$$

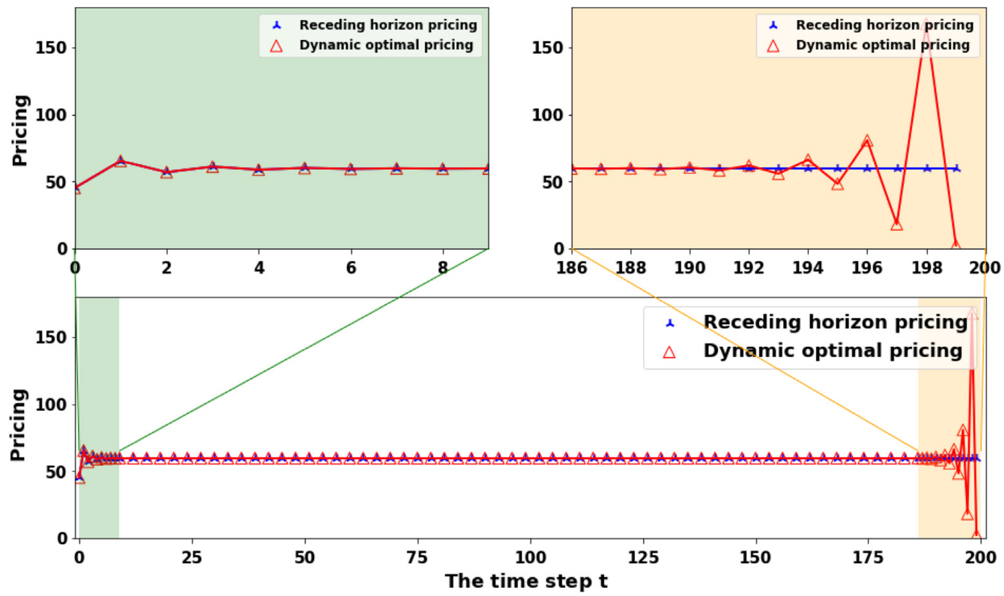

Fig. 5. The dynamic optimal pricing vs. the receding horizon pricing in Example 5.

$$e_{i,t+k|t} \leq a_{i,t+k|t} - h_i(\mathbf{u}_{i,t+k|t}), k \in \mathcal{K}; i \in \mathrm{V}, \tag{72}$$

$$\sum_{i=1}^{n} e_{i,t+k|t} = 0, k \in \mathcal{K}, \tag{73}$$

where $\mathbf{U}_{t\to t+N|t} = \{\mathbf{u}_{t|t}, \dots, \mathbf{u}_{t+N-1|t}\}$, $\mathbf{E}_{t\to t+N|t} = \{\mathbf{e}_{t|t}, \dots, \mathbf{e}_{t+N-1|t}\}$. Here $\mathbf{y}_{i,t+k|t}$ is the state of agent i at time step $t+k$ predicted at time step t. Similarly, $\mathbf{u}_{i,t+k|t}$ and $\mathbf{e}_{i,t+k|t}$ are the control action and trading decision of agent i at time step $t+k$ predicted at time step t obtained by starting from the current state $\mathbf{y}_{i,t|t} = \mathbf{y}_i(t)$ and applying to (71).

Let $\mathbf{U}^*_{t\to t+N|t} = \{\mathbf{u}^*_{t|t}, \dots, \mathbf{u}^*_{t+N-1|t}\}$, $\mathbf{E}^*_{t\to t+N|t} = \{\mathbf{e}^*_{t|t}, \dots, \mathbf{e}^*_{t+N-1|t}\}$ be the optimal solution of (70)–(73) and $\boldsymbol{\lambda}^*_{t\to t+N|t} = \{\lambda^*_{t|t}, \dots, \lambda^*_{t+N-1|t}\}$ be the optimal dual variables for constraints (73). The first element of $\mathbf{U}^*_{t\to t+N|t}$, $\mathbf{E}^*_{t\to t+N|t}$ and $\boldsymbol{\lambda}^*_{t\to t+N|t}$ is applied to the MAS-DALTD at time step t:

$$\mathbf{u}(t) = \mathbf{u}^*_{t|t}(y(t)), \tag{74}$$

$$\mathbf{e}(t) = \mathbf{e}^*_{t|t}(y(t)), \tag{75}$$

$$\lambda(t) = \lambda^*_{t|t}(y(t)). \tag{76}$$

Based on the new state $\mathbf{y}_{i,t+1|t+1} = \mathbf{y}_i(t+1), i \in \mathrm{V}$, the optimization problem (70)–(73) is solved repeatedly at time step $t+1$ and it yields a receding horizon control and pricing. The procedure of receding horizon control and pricing is summarized in Algorithm 1:

Algorithm 1 Receding horizon control and pricing

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while $t < T$ **do**
  measure the state $y(t)$ at time step t;
  obtain $\mathbf{U}^*_{t\to t+N|t}$, $\mathbf{E}^*_{t\to t+N|t}$, $\boldsymbol{\lambda}^*_{t\to t+N|t}$ by solving (70)–(73);
  apply the first element $\mathbf{u}^*_{t|t}$,$\mathbf{e}^*_{t|t}$, $\lambda^*_{t|t}$ to MAS-DALTD;

Note that Algorithm 1 still applies, even if the new optimization problem proposed in a receding horizon fashion over the horizon $[t, t+N]$ exceeds the entire time horizon $[0, T]$ when $t \geq T - N$.

Example 5. Consider a MAS-DALTD with the same setting in Example 4. Let the entire time horizon take the value of 200. We fix a prediction horizon as $N = 40$. First, we follow the procedure in Example 4 and compute the dynamic optimal pricing over the entire time horizon. Then we apply Algorithm 1 to obtain the receding horizon pricing. The resulting trajectories of the two pricing approaches are shown in Fig. 5.

From Fig. 5, it is clear that the dynamic optimal pricing and receding horizon pricing coincide over most of the time horizon, which shows that receding horizon pricing is a good approximation of the optimal pricing planned for the entire time horizon. We also note that the receding horizon pricing does not experience the large oscillations found in dynamic optimal pricing during the end periods of the time horizon. □

5. Conclusions

In this paper, we studied multi-agent systems with decentralized resource allocation without external resource supply. For multi-agent systems with static local allocation, we showed that under general concavity assumptions, the competitive equilibrium and the social welfare. Equilibrium exist and agree using a duality analysis. A major contribution of this paper is our formulation of a new social shaping problem for competitive equilibrium, where the optimal pricing is associated with an upper bound. We presented an explicit family of socially admissible utility functions under which the optimal pricing at a competitive equilibrium is always socially acceptable. Another important aspect of this paper is our study of dynamical multi-agent systems and we have generalized it in an optimal control context. We proved that the dynamic competitive equilibrium and social welfare equilibrium continue to exist and coincide with each other. In light of the dynamic programming concept, we also provided a recursive method of expressing and computing competitive equilibrium. We suggested a receding horizon strategy for smoothing dynamic pricing in order to shape it in the sense that the pricing trend will be stationary. Future work to construct a range of socially admissible utility functions in a generic way is possible. Future work to shape the optimal prices under the dynamic competitive equilibrium below a upper bound would also be an interesting direction.

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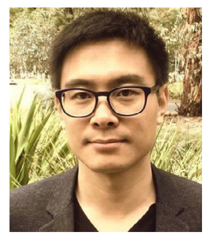

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