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Social Shaping of Linear Quadratic Multi-Agent Systems

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***Abstract*— In this paper, we study multi-agent systems with distributed resource allocation at individual agents. The agents make local resource allocation decisions including, in some cases, trading decisions — incurring income or expenditure subject to the resource price and system-level resource availability. The agents seek to maximize their individual payoffs, which accrue from both resource allocation income and expenditure. We define a social shaping problem for the system and show that the optimal price is always below a prescribed socially resilient price threshold. By exploring optimality conditions for each agent, we express resource allocation decisions in terms of piece-wise linear functions with respect to the price for unit resource. We further establish a tight range for the coefficients of the linear-quadratic utilities, under which optimal pricing is proven to be always socially resilient.**

I. Introduction

Recent technological advancements have enabled synthesized networked multi-agent systems (MAS) in both electric energy distribution systems and in automotive transportation. Agents in distributed and networked multi-agent systems have their own decisions, preferences, and objectives, and operate in concert with each other in order to achieve system-level objectives [1]–[6]. Efficient resource allocation is a fundamental problem for multi-agent systems, especially when demand must equal supply for efficient and secure system operation.

Insights from classical welfare economics theory [8], [9] have shown that it is possible to price resources in order to balance demand and supply in a market. In multi-agent systems with distributed resource allocations, agents decide on local resource consumption and exchange to optimize individual payoffs as a combination of local utility and income (or expenditure). Then a competitive equilibrium under resource pricing is reached when all agents maximize their individual payoffs subject to a network-level supply-demand balance, which in turn maximizes the overall system-level payoff [7].

The concept of operating a multi-agent system as a market via optimal pricing under a competitive equilibrium has applications in smart grid operations and climate-economy systems. In smart grids, agents represent households, and by optimally pricing energy we ensure the payoff for all households are maximized subject to the balance of energy supply and demand [12]–[18]. In climate-economy systems, agents represent countries, and optimal pricing of carbon emissions ensures the interests of each country are met subject to a carbon emission supply-demand balance [11], [19], [20]. However, in both cases, the optimally computed price is potentially not socially acceptable. For example, in February 2021, the price for electricity in Texas went to an unacceptably high rate after widespread power outages [23][1]. Moreover, the carbon emissions trading scheme under the Kyoto Protocol was widely criticized by researchers, as the estimated social cost of carbon was deemed as unfair among different regions [21], [22]. Consequently, it is important to prescribe the range of individual agent utility functions to ensure pricing under a competitive equilibrium is compatible with social norms, which motivates us to consider a social shaping problem for multi-agent systems extending our prior work in [10].

In this paper, we consider multi-agent systems where resources are distributed to individual agents, and agents make local demand and perhaps supply trading decisions to maximize their individual payoffs. The payoff decisions consider the summation of a utility arising from resource consumption and the income (or expenditure) from resource exchange. The agent utility functions take a linear-quadratic form, which passes through the origin. We formulate the social shaping problem for a competitive equilibrium at which the optimal pricing is always below a prescribed and socially resilient threshold. By exploring the optimality conditions of the agent payoff functions, we show that resource allocation decisions can be expressed in terms of piece-wise linear functions with respect to the price for a unit resource. Then, we link the monotonicity of these functions with the network-level demand-supply balance condition and establish a tight range for the coefficients in the linear-quadratic utilities, under which optimal pricing is proven to be always socially resilient and thus acceptable. In addition, we also show that the optimal load decisions and price under competitive equilibrium are equivalent with or without trading decisions when the price takes a positive value.

This paper is organized as follows. In Section II, we introduce our MAS and we introduce the social resilience problem. In Section III, we present our main results on social shaping of agent utility functions. In Section IV, we present numerical examples to validate each of the proposed three theorems introduced in Section III. Finally, Section V summarizes our contributions and considers opportunities for

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[1]Although the 2021 widespread power outage in Texas was considered an extremely rare ‘resilience’ event, climate change is expected to increase the frequency of weather-induced resilience events in power grids [25].

future work.

Notation: $\mathbb{R}$ and $\mathbb{R}^{\geq 0}$ represent the set of real numbers and non-negative real numbers, respectively. $\mathbb{R}^{t}_{>0}$ denotes vectors of dimension t whose elements are positive real numbers.

II. Problem Definition

In this section, we define our MAS and we introduce our social resilience problem.

A. MAS with Static Load Decisions

Consider n agents indexed in the set $V = \{1, 2, ..., n\}$. Let $a_i \in \mathbb{R}^{\geq 0}$ be the local resource generated by agent i, and $x_i \in \mathbb{R}^{\geq 0}$ be the resource consumed by this agent. Let $f_i(x_i) : \mathbb{R}^{\geq 0} \mapsto \mathbb{R}$ denote the utility of each agent as a result of consuming x_i amount of resource. The difference between x_i and a_i will be exchanged over the MAS through an underlying network. The network capacity $C > 0$ is defined as $C := \sum_{i=1}^{n} a_i$.

The aim of the MAS is to allocate the total network capacity C over the x_i under a price λ for unit resource exchange, where each agent tries to maximize its individual payoff function as the summation of the utilities and the income (or expenditure). Such an MAS is termed an *MAS with Static Load Decisions (MAS-SLD).*

Let $\mathbf{a} = (a_1, ..., a_n)^\top$ be the vector of all resources, and let $\mathbf{x} = (x_1, ..., x_n)^\top$ be the vector of consumed resources. We recall the following definition [16].

Definition 1: For the MAS-SLD, a competitive equilibrium (λ^*, $\mathbf{x}^*$) is achieved if the following two conditions hold:

(i) $\mathbf{x}^*$ maximizes the individual payoff function of each agent; i.e., x_i^* solves the following maximization problem

$$\begin{aligned} \max_{x_i} \quad & f_i(x_i) + \lambda^*(a_i - x_i) \\ \text{s.t.} \quad & x_i \in \mathbb{R}^{\geq 0}, \end{aligned} \tag{1}$$

(ii) $\mathbf{x}^*$ balances the total energy consumption and supply across the network; that is,

$$\sum_{i=1}^{n} x_i^* = C. \tag{2}$$

B. MAS with Static Load and Trading Decisions

In order to extend the MAS-SLD, let $e_i \in \mathbb{R}$ represent the amount of resource traded by agent i. It is clear that e_i can never be greater than $a_i - x_i$. Now, each agent has two associated decision variables x_i and e_i. Such an MAS is termed an *MAS with Static Load and Trading Decisions (MAS-SLTD).* Let $\mathbf{e} = (e_1, ..., e_n)^\top$ be the vector of traded resource across the network.

Definition 2: A competitive equilibrium $(\lambda^*, \mathbf{x}^*, \mathbf{e}^*)$ for an MAS-SLTD is achieved if the following two conditions hold:

(i) $(\mathbf{x}^*, \mathbf{e}^*)$ maximizes the individual payoff function of each agent, meaning that (x_i^*, e_i^*) solves the following maximization problem

$$\begin{aligned} \max_{x_i, e_i} \quad & f_i(x_i) + \lambda^* e_i \\ \text{s.t.} \quad & x_i + e_i \leq a_i, \qquad x_i \in \mathbb{R}^{\geq 0},\ e_i \in \mathbb{R}, \end{aligned} \tag{3}$$

(ii) $\mathbf{e}^*$ balances the total resource consumption and supply across the network, i.e.,

$$\sum_{i=1}^{n} e_i^* = 0. \tag{4}$$

C. Social Optimality of Competitive Equilibrium

From classical welfare economics theory, a competitive equilibrium guarantees Pareto optimality in the sense that no agent can update her decision without reducing the payoff of other agents. This is referred to as market efficiency [7]–[9]. One may also introduce the following system-level social welfare optimization problem for an MAS-SLD as

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i=1}^{n} f_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = C, \qquad x_i \in \mathbb{R}^{\geq 0},\ i \in V. \end{aligned} \tag{5}$$

It is known that under the concavity of the f_i for an MAS-SLD, the optimal resource allocation decision in a competitive equilibrium is an optimal solution to this social welfare optimization problem, and vice versa [10], [16]. In addition, the optimal price λ^* in (1) is the Lagrange multiplier associated with the equality constraint $\sum_{i=1}^{n} x_i = C$ in (5). The same connection between competitive equilibrium and social welfare optimization can be drawn for an MAS-SLTD.

D. Social Shaping for Linear Quadratic MAS

Despite the aforementioned efficiency and social optimality of the competitive equilibrium for our MAS, one critical challenge arises in its practical usefulness. Specifically, if a subset of agents potentially set their utility functions in aggressive ways, they may dominate the optimal system price λ^* so that the price becomes unaffordable for other agents. Therefore, it is important to set a range for the agent utility functions so that the resulting λ^* is always below a socially acceptable threshold $\lambda^\dagger > 0$ [10]. Here, we assume $\lambda^\dagger$ is designed by MAS-based operator in consultation with respective agents. To this end, we introduce the following assumption, which is assumed to hold throughout the rest of the paper.

Assumption 1: For all $i \in V$,

$$f_i(x_i) := -\frac{1}{2} b_i x_i^2 + k_i x_i = -\frac{1}{2} b_i x_i^2 + m_i b_i x_i. : \mathbb{R}^{\geq 0} \mapsto \mathbb{R}, \tag{6}$$

where $(b_i, k_i) \in \mathbb{R}^{>0} \times \mathbb{R}^{>0}$, and $m_i = k_i / b_i$.

The social shaping problem for these linear quadratic utility functions is presented in the following as an extension of the study in [10].

MAS Social Shaping Problem. For both an MAS-SLD and an MAS-SLTD, find the range of values for $(m_{\max}, b_{\max})$ under which all utility functions with parameters $m_i \leq m_{\max}$ and $b_i \leq b_{\max}$ will lead to a socially resilient optimal price $\lambda^* \leq \lambda^\dagger$.

III. Main Results

In this section, we present a new set of utility functions for an MAS such that the optimal price is socially resilient.

A. MAS with Static Load Decisions

We first consider an MAS-SLD quadratic utility function of the form in (6).

Lemma 1: Considering the utility function $f_i(x_i) = -\frac{1}{2}b_i x_i^2 + m_i b_i x_i$ in (6) and the optimization problem (1), the optimal solution x_i^* is such that

$$x_i^* = \max\left\{m_i - \frac{\lambda^*}{b_i}, 0\right\}. \tag{7}$$

Proof: Rearranging the optimization problem (1), x_i^* is the solution to the following maximization problem:

$$\begin{aligned} \max_{x_i} \quad & -\frac{1}{2}b_i x_i^2 + (m_i b_i - \lambda^*)x_i + \lambda^* a_i \\ \text{s.t.} \quad & x_i \in \mathbb{R}^{\geq 0}. \end{aligned} \tag{8}$$

Let $\hat{x}_i$ be the value x_i which maximizes the objective function in the absence of any constraints. Then $\hat{x}_i$ is obtained when the derivative of the objective function equals zero. That is,

$$-b_i\hat{x}_i + (m_i b_i - \lambda^*) = 0, \tag{9}$$

which implies $\hat{x}_i = m_i - \frac{\lambda^*}{b_i}$. Considering the inequality constraint $x_i \geq 0$ in the maximization problem (8), when $\lambda^* \leq m_i b_i$, the solution is achieved at $x_i^* = \hat{x}_i = m_i - \frac{\lambda^*}{b_i}$ which is non-negative and satisfies the inequality constraint. Conversely, when $\lambda^* > m_i b_i$, $\hat{x}_i$ is negative which does not satisfy the inequality constraint, so $x_i^* \neq \hat{x}_i$. In this case, the objective function is strictly decreasing with respect to x_i. Consequently, in order for the objective function to be maximized, x_i must be minimized, i.e., $x_i^* = 0$. Therefore, when $\lambda^* > m_i b_i$ then $x_i^* = 0$, otherwise, $x_i^* = m_i - \frac{\lambda^*}{b_i}$. ■

Lemma 2: If $m_i \leq \frac{C}{n}$ for all $i \in V$, then $\lambda^* \leq 0$. Conversely, if $m_i > \frac{C}{n}$ for all $i \in V$, then $\lambda^* > 0$.

Proof: (i) Consider the case $m_i \leq \frac{C}{n}$. By contradiction, suppose $\lambda^* > 0$. From equation (7) we yield $x_i^* < m_i$. Since $m_i \leq \frac{C}{n}$, we obtain $x_i^* < \frac{C}{n}$. Consequently,

$$\sum_{i=1}^{n} x_i^* < C, \tag{10}$$

which contradicts the balancing equality $\sum_{i=1}^{n} x_i^* = C$ in (2). Therefore, it follows that $\lambda^* \leq 0$.
(ii) Consider the case $m_i > \frac{C}{n}$. By contradiction, suppose $\lambda^* \leq 0$. From equation (7) we obtain $x_i^* \geq m_i$. Since $m_i > \frac{C}{n}$, we yield $x_i^* > \frac{C}{n}$. Consequently,

$$\sum_{i=1}^{n} x_i^* > C, \tag{11}$$

which contradicts the balancing equality $\sum_{i=1}^{n} x_i^* = C$ in (2). Therefore, it follows that $\lambda^* > 0$. ■

Now consider two vectors $k = (k_1, ..., k_n)$ and $k' = (k_1', ..., k_n')$, and let $k \preceq k'$ denote $k_i \leq k_i'$ for all $i \in V$. Let $m = (m_1, ..., m_n)$ and $b = (b_1, ..., b_n)$. Suppose λ^* is the optimal price associated with the pair of vectors (m, b), and let $\lambda^{*'}$ be the optimal price associated with (m', b').

Lemma 3: If $\lambda^* > 0$, then $m \preceq m'$ and $b \preceq b'$ yield $\lambda^* \leq \lambda^{*'}$.

Proof: Suppose $\lambda^* > 0$. Substituting (7) into the balancing equality $\sum_{i=1}^{n} x_i^* = C$ in (2) yields

$$\sum_{i=1}^{n} \max\left\{m_i - \frac{\lambda^*}{b_i}, 0\right\} = C. \tag{12}$$

As m_i and b_i increase, λ^* must also increase so as to compensate for the change — ensuring the balancing equality (12) holds. Otherwise, the left-hand side of equality (12) would increase, while the right-hand side remains constant, and so the equality would not hold. ■

Theorem 1: Consider the MAS-SLD. Suppose $(m_{\max}, b_{\max}) \in \mathbb{R}^2_{>0}$ is selected from the following set

$$\begin{aligned} \mathscr{S}_* = & \left\{m_{\max} \leq \frac{C}{n},\, b_{\max} \in \mathbb{R}^{>0}\right\} \bigcup \\ & \left\{m_{\max} > \frac{C}{n},\, b_{\max} \leq \frac{n\lambda^\dagger}{n m_{\max} - C}\right\}, \end{aligned} \tag{13}$$

then the resulting λ^* is socially resilient for all utility functions satisfying $m_i \leq m_{\max}$ and $b_i \leq b_{\max}$.

Proof: We investigate two cases.

Case (i) $m_{\max} \leq \frac{C}{n}$. In this case, $m_i \leq m_{\max}$ implies $m_i \leq \frac{C}{n}$ for $i \in V$. Therefore, Lemma 2 implies $\lambda^* \leq 0$. Since $\lambda^\dagger > 0$, one obtains $\lambda^* < \lambda^\dagger$. Consequently, λ^* is socially resilient.

Case (ii) $m_{\max} > \frac{C}{n}$ and $b_{\max} \leq \frac{n\lambda^\dagger}{n m_{\max} - C}$. If $\lambda^* \leq 0$, then it is socially resilient. Conversely, if $\lambda^* > 0$, Lemma 3 yields λ^* is monotonically increasing with respect to m_i and b_i, so the highest possible price $\lambda^*_{\max}$ is achieved when $m_i = m_{\max}$ and $b_i = b_{\max}$ for all agents $i \in V$. Consequently, when all agents select $m_i = m_{\max}$ and $b_i = b_{\max}$, the balancing equality (12) results in

$$n\left(m_{\max} - \frac{\lambda^*_{\max}}{b_{\max}}\right) = C, \tag{14}$$

and therefore,

$$\lambda^*_{\max} = b_{\max}\left(\frac{n m_{\max} - C}{n}\right). \tag{15}$$

From equation (15), along with the assumption $b_{\max} \leq \frac{n\lambda^\dagger}{n m_{\max} - C}$ in (13), yields $\lambda^*_{\max} \leq \lambda^\dagger$. Since $\lambda^* \leq \lambda^*_{\max}$, one obtains $\lambda^* \leq \lambda^\dagger$.

Considering (i) and (ii), it follows that as long as $(m_{\max}, b_{\max})$ is constrained in the set $\mathscr{S}_*$ in (13), λ^* will be socially resilient. ■

B. MAS with Static Load and Trading Decisions

This part studies the MAS-SLTD.

Lemma 4: Consider the utility function $f_i(x_i) = -\frac{1}{2}b_i x_i^2 + m_i b_i x_i$ in (6) and the optimization problem (3). The optimal solution x_i^* is achieved by

$$x_i^* = \max\left\{m_i - \frac{\lambda^*}{b_i}, 0\right\}. \tag{16}$$

Proof: Consider the inequality constraint $x_i + e_i \leq a_i$ in the optimization problem (3). This inequality can be written as

$$x_i + e_i + s_i = a_i, \tag{17}$$

where $s_i \geq 0$ is the slack variable. Substituting $e_i = a_i - x_i - s_i$ into (3) yields an equivalent form for the optimization problem as

$$\begin{aligned} \max_{x_i, s_i} \quad & -\frac{1}{2}b_i x_i^2 + (m_i b_i - \lambda^*)x_i + \lambda^* a_i - \lambda^* s_i \\ \text{s.t.} \quad & x_i, s_i \in \mathbb{R}^{\geq 0}. \end{aligned} \tag{18}$$

Furthermore, it is proved that for systems with load and trading decisions $\lambda^* \geq 0$ [10]. Therefore, we can consider three cases.

Case (i) $\lambda^* = 0$. In this case, (18) yields $x_i^* = m_i$.

Case (ii) $0 < \lambda^* \leq m_i b_i$. In this case, the objective function in (18) is strictly decreasing with respect to s_i. Consequently, in order for the objective function to be maximized, s_i must be minimized, i.e., $s_i^* = 0$. This implies that the first inequality constraint in (3) is active. As a result, substituting $s_i^* = 0$ into (18) yields

$$\begin{aligned} \max_{x_i} \quad & -\frac{1}{2}b_i x_i^2 + (m_i b_i - \lambda^*)x_i + \lambda^* a_i \\ \text{s.t.} \quad & x_i \in \mathbb{R}^{\geq 0}. \end{aligned} \tag{19}$$

This optimization problem is the same as the one for systems with only load decisions, described in (8). Therefore, considering $\lambda^* \leq m_i b_i$, Lemma 1 implies $x_i^* = m_i - \frac{\lambda^*}{b_i}$.

Case (iii) $\lambda^* > m_i b_i$. In this case, the objective function in (18) is strictly decreasing with respect to both x_i and s_i. Consequently, in order for the objective function to be maximized, x_i and s_i must be minimized, i.e., $x_i^* = s_i^* = 0$.

Finally, considering (i), (ii), and (iii), it follows that (16) holds. ■

Lemma 5: For all $i \in V$, if $m_i \leq \frac{C}{n}$ then $\lambda^* = 0$. Conversely, if $m_i > \frac{C}{n}$ then $\lambda^* > 0$.

Proof: We use a similar proof to the proof of Lemma 3. ■

Theorem 2: Consider the MAS-SLTD. Suppose $(m_{\max}, b_{\max}) \in \mathbb{R}^2_{>0}$ is selected from the set

$$\begin{aligned} \mathscr{S}_* = &\left\{m_{\max} \leq \frac{C}{n},\, b_{\max} \in \mathbb{R}^{>0}\right\} \bigcup \\ &\left\{m_{\max} > \frac{C}{n},\, b_{\max} \leq \frac{n\lambda^\dagger}{n m_{\max} - C}\right\}, \end{aligned} \tag{20}$$

then the resulting λ^* is always socially resilient.

Proof: We consider two cases.

Case (i) $m_{\max} \leq \frac{C}{n}$. In this case, $m_i \leq m_{\max}$ implies $m_i \leq \frac{C}{n}$ for $i \in V$. Therefore, Lemma 5 implies $\lambda^* = 0$. Since $\lambda^\dagger > 0$, we obtain $\lambda^* < \lambda^\dagger$. Consequently, λ^* is socially resilient.

Case (ii) $m_{\max} > \frac{C}{n}$ and $b_{\max} \leq \frac{n\lambda^\dagger}{nm - C}$. In this case $m_i \leq m_{\max}$, and either $m_i \leq \frac{C}{n}$ or $m_i > \frac{C}{n}$. It follows that either $\lambda^* = 0$ or $\lambda^* > 0$. First, if $\lambda^* = 0$, then it is socially resilient. Conversely, if $\lambda^* > 0$, the objective function in (18) is strictly decreasing with respect to s_i, hence $s_i^* = 0$. Therefore, (17) yields $x_i^* + e_i^* = a_i$, i.e., the inequality constraint is active. Considering $\sum_{i=1}^n e_i^* = 0$, it follows that

$$\sum_{i=1}^n x_i^* = C, \tag{21}$$

where $C = \sum_{i=1}^n a_i$. Substituting equation (16) into (21) yields

$$\sum_{i=1}^n \max\left\{m_i - \frac{\lambda^*}{b_i}, 0\right\} = C. \tag{22}$$

The rest of the proof is the same as the proof of Theorem 1 part (ii). Considering equality (22) and $\lambda^* > 0$, Lemma 3 implies λ^* is monotonically increasing with respect to m_i and b_i, so the highest possible optimal price $\lambda^*_{\max}$ is achieved when $m_i = m_{\max}$ and $b_i = b_{\max}$ for all agents. Consequently, when all agents select $m_i = m_{\max}$ and $b_i = b_{\max}$, the equality (22) results in

$$n\left(m_{\max} - \frac{\lambda^*_{\max}}{b_{\max}}\right) = C, \tag{23}$$

and therefore,

$$\lambda^*_{\max} = b_{\max}\left(\frac{n m_{\max} - C}{n}\right). \tag{24}$$

From equation (24), along with the assumption $b_{\max} \leq \frac{n\lambda^\dagger}{n m_{\max} - C}$ in (20), yields $\lambda^*_{\max} \leq \lambda^\dagger$. Since $\lambda^* \leq \lambda^*_{\max}$, we obtain $\lambda^* \leq \lambda^\dagger$.

Considering (i) and (ii), it follows that as long as $(m_{\max}, b_{\max})$ is constrained in the set $\mathscr{S}_*$ in (20), λ^* will be socially resilient. ■

C. Decision Equivalence

Theorem 3: Suppose $\lambda^* > 0$. Then under a competitive equilibrium, MAS-SLD and MAS-SLTD are equivalent in that each agent specifies the same load allocation and the optimal price is also the same.

Proof: Consider the optimization problem of MAS-SLTD in (3). The inequality constraint $x_i + e_i \leq a_i$ can be written as $x_i + e_i + s_i = a_i$, where $s_i \geq 0$ is the slack variable. Additionally, substituting $e_i = a_i - x_i - s_i$ into (3) yields an equivalent form for the optimization problem as

$$\begin{aligned} \max_{x_i, s_i} \quad & f_i(x_i) + \lambda^*(a_i - x_i - s_i) \\ & x_i, s_i \in \mathbb{R}^{\geq 0}. \end{aligned} \tag{25}$$

Let $\lambda^* > 0$. Then, the objective function in (25) is strictly decreasing with respect to s_i. Consequently, in order for (25) to be maximized, s_i must be minimized, i.e., $s_i^* = 0$. This implies that the inequality constraint $x_i + e_i \leq a_i$ is active and $x_i^* + e_i^* = a_i$. Taking the summation in this equality implies $\sum_{i=1}^{n} x_i^* + \sum_{i=1}^{n} e_i^* = C$, where $C = \sum_{i=1}^{n} a_i$. Substitution of the balancing equality $\sum_{i=1}^{n} e_i^* = 0$ in (4), yields

$$\sum_{i=1}^{n} x_i^* = C. \tag{26}$$

Furthermore, substituting $s_i = 0$ into (25) yields

$$\begin{aligned} \max_{x_i} \quad & f_i(x_i) + \lambda^*(a_i - x_i) \\ & x_i \in \mathbb{R}^{\geq 0}. \end{aligned} \tag{27}$$

Comparing (27) and (26) with (1) and (2), implies that the problem of MAS-SLTD is equivalent to the problem of MAS-SLD. Therefore, $x_{SLTD}^* = x_{SLD}^*$. A similar analysis can be done to show $\lambda_{SLTD}^* = \lambda_{SLD}^*$. ■

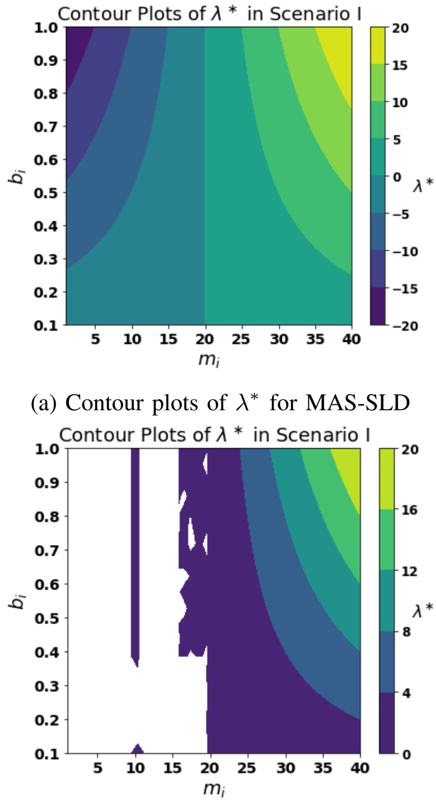
IV. Numerical Results

Consider a MAS with 4 agents. Let $a = (10, 21, 24, 25)$, which yields $C = 80$. The highest socially acceptable resource price is $\lambda^\dagger = 20$, and thus we seek a price $\lambda^* \leq 20$. Under Assumption 1, each agent has the utility function $f_i(x_i) = -\frac{1}{2} b_i x_i^2 + m_i b_i x_i$.

In the numerical results that follow we consider MAS-SLD and MAS-SLTD. To calculate λ^* for MAS-SLD, the social welfare problem in (5) must be solved, where λ^* is the Lagrange multiplier associated with the equality constraint in (5). Similarly, to obtain λ^* for MAS-SLTD, the associated social welfare problem must be solved, where λ^* corresponds to the Lagrange multiplier associated with the balancing equality constraint. In the following, we examine four scenarios.

Scenario I: For simplification, suppose all agents have the same m_i and the same b_i which take values in the intervals $[1, 40]$ and $[0.1, 1]$, respectively. Contour plots of λ^* as a function of m_i and b_i are depicted in Fig. 1 for both MAS-SLD and MAS-SLTD. In this case, we have $m_{\max} = 40$ and $b_{\max} = 1$ which lie in the proposed set $\mathscr{S}_*$ defined in Theorems 1 and 2. As Fig. 1 illustrates, for different values of m_i and b_i, the optimal price is socially resilient, i.e., $\lambda^* \leq 20$. This is because $(m_{\max}, b_{\max})$ is inside the prescribed set, and therefore, Theorems 1 and 2 are valid. In addition, when $\lambda^* > 0$, i.e., $m_i > 20$, the optimal price is monotonically increasing with respect to both m_i and b_i, which is consistent with Lemma 3.

Scenario II: In the second scenario, consider $m_{\max} = 40$ and $b_{\max} = 2$, which exceeds the proposed set $\mathscr{S}_*$. Suppose all agents have the same m_i which takes values in the interval $[1, 40]$, while b_i is set according to $b = (1.2, 1.3, 1.5, 1.8)$. λ^* is calculated for different values of $m_i \in [1, 40]$, and the results are depicted in Fig. 2 for both MAS-SLD and MAS-SLTD. Denote λ_{SLD}^* and λ_{SLTD}^* as the optimal prices of MAS-SLD and MAS-SLTD, respectively. Fig. 2 illustrates that for $m_i > 35$, the optimal price is greater than $\lambda^\dagger$, and therefore, it is not socially resilient. This happens because $b_{\max}$ does not follow the conditions of Theorems 1 and 2, and it is not in the set $\mathscr{S}_*$. Additionally, it is observed that when $m_i \leq \frac{C}{n}$ then $\lambda^* \leq 0$ for MAS-SLD and $\lambda^* = 0$ for MAS-SLTD. However, when $m_i > \frac{C}{n}$ then $\lambda^* > 0$ for



(a) Contour plots of λ^* for MAS-SLD

(b) Contour plots of λ^* for MAS-SLTD

Fig. 1: Contour plots of λ^* as a function of m_i and b_i for MAS-SLD and MAS-SLTD in Scenario I.

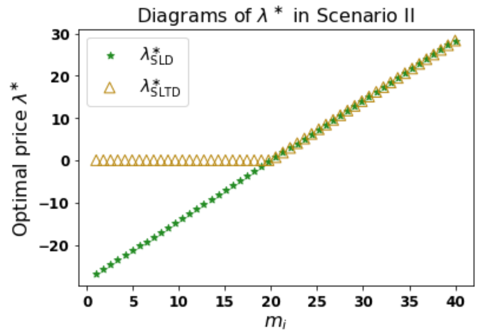


Fig. 2: Diagrams of the optimal price λ^* as a function of m_i in Scenario II.

both MAS-SLD and MAS-SLTD, which is consistent with Lemmas 2 and 5 (note that $\frac{C}{n} = 20$).

Scenario III: In the third scenario, suppose $m_{\max} = 40$ and $b_{\max} = 1$ which lie in the proposed set. We study two cases. In case (1), let $m_i = m_{\max}$ and in case (2) let m_i be set according to $m = (10, 14, 17, 20)$. Denote $\lambda_{\text{SLD}}^{(1)*}$ and $\lambda_{\text{SLD}}^{(2)*}$ as the optimal prices of MAS-SLD associated with cases (1) and (2), respectively. Similarly, let $\lambda_{\text{SLTD}}^{(1)*}$ and $\lambda_{\text{SLTD}}^{(2)*}$ be the optimal prices of MAS-SLTD related to cases (1) and (2), respectively. Suppose all agents have the same b_i. The diagrams of λ^* as a function of b_i are depicted in Fig. 3 when b_i takes values in the interval $[0.1, 2]$. As Fig. 3

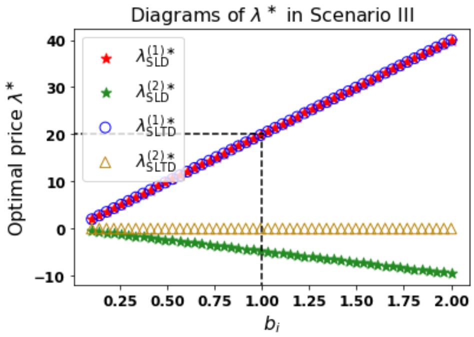


Fig. 3: Diagrams of the optimal price λ^* as a function of b_i in Scenario III.

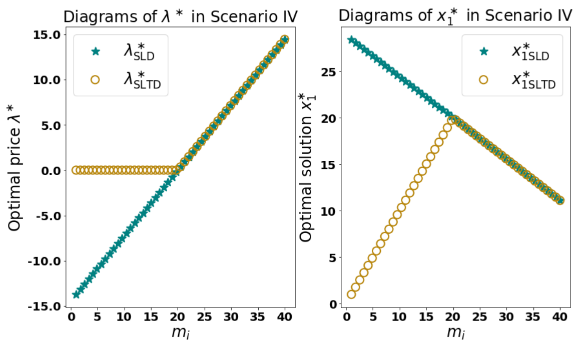


Fig. 4: Diagrams of λ^* and x_1^* as a function of m_i in Scenario IV.

illustrates, in case (1), λ^* is socially resilient only if $b_i \leq 1$ (note that $b_{\max} = 1$). On the other hand, in case (2), where $m_i \leq \frac{C}{n}$, λ^* is socially resilient for different values of b_i, no matter if $b_i \leq 1$ or $b_i > 1$. This validates Theorems 1 and 2.

Scenario IV: Consider the system parameters as $b = (0.5, 0.7, 0.9, 1)$ and $m_i \in [1, 40]$, where all agents have the same m_i. To assess Theorem 3, diagrams of λ^* and x_1^* as a function of m_i are depicted in Fig. 4 for both MAS-SLD and MAS-SLTD. As illustrated in Fig. 4, for $m_i > 20$, where $\lambda^* > 0$, we have $x^*_{1SLTD} = x^*_{1SLD}$ and $\lambda^*_{SLTD} = \lambda^*_{SLD}$.

V. Conclusion

In this paper, we presented the problem of social shaping for self-sustained multi-agent systems with quadratic utility functions and distributed resource allocation, in which each agent aims to maximize its payoff under quadratic utility functions. It was shown that the resulting optimal solution is piece-wise linear with respect to the optimal price. Based on this observation, we presented a set of quadratic utility functions for these agents which guarantees that the optimal price across the network will always be socially acceptable. Subsequently, a few numerical examples are provided to examine the effectiveness of the proposed set. In future work, extensions to more general classes of utility functions is possible. Additionally, extension to consider the payoff function of each agent to be dependent on the resource allocation of other agents, resulting in more complicated objective functions, may also be possible. Other extensions may include MAS dynamics, and the impact of price volatility.

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